GENERAL COLLEGE PHYSICS

RANDALL
WILLIAMS
COLBY

Coefficient of Expansion

Steel	.000012
Brass	.000019
Hg	.00018

Cost me $1.50

W. R. Johnson
20 Cloudman

I name of exp
II obj of exp
III apparatus
IV Diagrams
V Theory + Discussion of procedure
VI Data sheet
VII calculations
VIII Exercises + Conclusions

~~H. B. Ware~~

I_5 ~~[illegible]~~

F_3 no

~~Coop 57~~

I_2

X_3 good ~~121 N. Howard St.~~

L_2 good ~~Decatur, Ga.~~

H_4

F_1 Ewalt

F_2 dynamo analysis

S_1 ~~Beatrice Cannon~~

L_3 ~~729 Grant St~~

~~Atlanta Ga~~

80
60

15

2.997

$\frac{E_1^2}{R_1}$ $\frac{E_2^2}{R_2}$ $\frac{E^2}{1}$

$\frac{E^2}{\rho\frac{l}{a_1}}$ = $\frac{E_2^2}{\rho\frac{l}{a_2}}$

B

Indentifaction and Indication of
sea coast targets
chapter XIII page 176 to 184 incl
" XI " 208 " 217 " 5th Ed

GENERAL COLLEGE PHYSICS

GENERAL
COLLEGE PHYSICS

By

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GENERAL
COLLEGE PHYSICS

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PREFACE

The authors have endeavored in this book to present material which is suitable to form a foundation for serious work in the physical sciences without losing sight of the needs of those students who take a course in Physics for its cultural value rather than as a preparation for future work. The discipline which comes from a careful study of the classic principles of the subject is indispensable to a grasp of any of its modern phases. Mechanics in particular contains a well-organized body of concepts which underlie most work not only in Physics but in the cognate sciences. For this reason this portion of the book has been given ample proportions and has been accompanied by a large number of problems which should serve to make the principles and operations familiar and fluent.

Nevertheless, it has been our endeavor to write even the classic portions of the subject from a modern point of view, so that no obsolete phraseology need be acquired which later will hinder the less restricted outlook of the newer work. For example, in the sections on Electricity and Magnetism an attempt has been made to intermingle the electron theory in such a manner that the student will feel that it is not a different part of the subject and that the older formulation will never lose its clarity and usefulness no matter how much the modern analysis may succeed in furnishing detail. Without this emphasis on the classic work, modern theories can easily give a superficial and speculative impression.

In the section on Light some space has been devoted to more recent spectral theories. The hydrogen spectrum has been discussed somewhat at length. On the other hand, such subjects as Relativity have been omitted because it has been believed that students at this stage are not prepared to undertake this as a serious study and that a popular and superficial presentation might easily work harm. Where concepts are so intricate that they cannot be handled without elaborate mathematical treatment, the young student is either bewildered or led into superficial habits of thought which have no place in scientific work.

This text has been used in the form of mimeographed pages for a number of semesters in the University of Michigan, and it has been possible as a result to remove all material which has not been found essential or teachable. During this period the authors have profited from the criticisms and suggestions of several members of the staff of the Physics Department, and they welcome this opportunity of expressing their appreciation of the assistance thus received. The endeavor has been to produce a book containing only material suitable for instruction in the course and of such a length that it can be covered in a year's work.

Ann Arbor, Oct. 1, 1929

H. M. R.
N. H. W.
W. F. C.

MECHANICS
OF SOLIDS

2 mi = 1 km
53.6 gm = 1 lb.

CHAPTER I

UNITS AND MEASUREMENT

1. Introduction.—The physical sciences are concerned primarily with the phenomena of inanimate nature, while the biological sciences deal with those natural phenomena which are related to the organic or animate world. When we come to appreciate that most of the processes involved in the life of the cell are either physical or chemical, we realize that the fields of these two classes of science are by no means distinct. Similarly, when we consider the different branches of physical science, as physics, chemistry, astronomy, geology, etc., we find it impossible to define accurately the limits of each. Physics deals with phenomena, laws, and processes that are fundamental to each of the other branches of physical science. Astronomy is largely devoted to the application of the principles and methods of physics to the special problems arising from the study of the celestial bodies; in fact, a branch of astronomy is now known by the name *astrophysics*, and the term *celestial mechanics* is frequently met in the literature of the subject. Formerly it was customary to regard all questions of the ultimate constitution of matter as chemical problems, but in recent years molecular structure and the constitution of the atom are among the most important research problems in physics. The successful engineer is the man who knows physics and how to apply it. The physician and surgeon and the experimental psychologist constantly find use for their knowledge of physics, and often make use of modern physical technique.

2. Measurement.—Physics is frequently referred to as one of the exact sciences. This expresses the idea that physics depends upon accurate measurement as a means of attaining its ends. Since this is the case, it is evident that at the beginning we must lay the foundation for the system of units that will be found necessary as we proceed with the subject. Measurement implies the comparison of a physical quantity with another quantity of the same kind which is adopted as a standard or unit. The representation of the physical quantity is always in terms of the unit used in its measurement, together with the number expressing how many times larger it is than the unit. Thus a distance might be expressed as five miles. The number without the name of the unit would give no idea of the distance. It is found that if we fix upon three independent physical quantities and consider them as fundamental, all other quantities related to mechanics may be defined in terms of these three. In the system of units that is universally used, length, mass, and time are considered as fundamental quantities, and the units of length, mass, and time are called *fundamental units*. The other units are called *derived units* because they measure quantities that are related to one or more of the fundamental quantities and are themselves expressed in terms of the fundamental units. The unit of area is a derived unit, since it is merely the square described on the unit of length. Speed is defined as the time rate at which a body travels. It may be expressed in miles per hour; hence only the length unit and the time unit are involved in the definition of the unit of speed.

3. Units of Length.—In the metric system the unit of length is the meter, which is the distance between two marks on a certain platinum-iridium bar, provided the bar is at the temperature of melting ice. This bar is known as the *international meter* and is kept at the International Bureau of Weights and Measures near Paris. The centimeter is 1/100 part of the meter, and the millimeter is 1/1,000 part of the meter.

In the English system the unit of length is the yard. In the United States it is defined by Act of Congress as 3,600/3,937 of a meter.

4. Unit of Time.—The second is the unit of time in both the metric and the English systems. It is defined as 1/86,400 part of a mean solar day. The number 86,400 is the product of 60, 60 and 24. A mean solar day is specified because the length of the solar day varies slightly in different seasons of the year. The length of the solar day is the interval of time between two successive passages of the center of the sun across a meridian. This interval depends principally upon the speed of rotation of the earth upon its axis, but it also depends to a slight extent upon the speed of revolution of the earth in its orbit. Since the speed of the earth in its orbit is not constant for all parts of the orbit, it is evident that the length of the solar day must vary slightly. In Fig. 1, let the ellipse represent the orbit of the earth and S the position of the center of the sun. S is one of the foci of the ellipse. Let E represent the position of the earth at a given time and E' its position about one day later. The sun is on the meridian for an observer at m; and in order that it may be on the meridian again the next day for the same observer, the earth must have turned, in the counterclockwise sense, through 360° plus the angle $m'oS$. The size of the angle $m'oS$ is determined by the distance the earth moves along the orbit during the day and also by the distance between the earth and the sun. The speed of the earth in its orbit is greater when the earth is near the sun than when it is farther away; hence the length of the solar day is greater when the earth is in the neighborhood of A than when it is on the opposite side of its orbit.

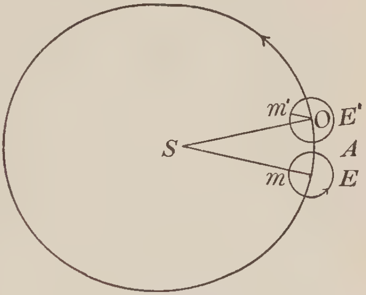


FIG. 1

5. Unit of Mass.—Among the universal properties of matter, inertia is perhaps the most familiar. It is defined as that property of matter whereby it resists any attempt to change its velocity. It is difficult to put a freight train in motion largely because of its inertia, and it is likewise difficult to stop an automobile on a slippery pavement because of its inertia. Any attempt to change the motion of a body meets with resistance. The measure of the inertia of a body is called its mass. The amount of muscular exertion necessary to set a body in motion gives some idea of the amount of matter in it; one can tell whether a barrel is full or empty by moving it, but accurate comparisons of masses cannot be made in this way. We must avoid the use of the muscular sense since it cannot be trusted. Let us assume two bodies to be given equal speeds toward each other in the same straight line. It is not difficult to arrange conditions such that the bodies will collide without rebound. If both bodies are found to be at rest after the collision, they have suffered equal changes of velocity, there have been equal resistances to the change on the part of each; thus they have equal inertias and hence equal masses. The weight of a body is the force which the earth exerts upon it pulling it vertically downward. All bodies near the earth have weight, but the same body does not have the same weight at all places; it does, however, have the same inertia at all places. We might agree to consider two masses as being equal, provided they have the same weight at a given place. The comparison might be made by means of a beam balance with equal arms, granting for the present that such a balance could be made. Whether these two arbitrarily chosen methods of measuring mass will give the same result can only be determined by experiment. As a matter of experience, they do; and so the process of weighing, being simple and accurate, takes the place of the more difficult process of comparing masses by comparing their inertias.

The *gram* is the unit of mass in the metric system. It is the 1/1,000 part of the mass of the international kilogram which is kept at the International Bureau of Weights and Measures near Paris. It is very nearly equal to the mass of a cubic centimeter of distilled water at 4° centigrade.

The *pound* is the unit of mass in the English system. As used in the United States, its definition is given in terms of the kilogram: 1 kilo = 2.2046 lb. This definition was made legal by Act of Congress in 1866.

6. The Absolute System of Units.—The system of units built upon the three fundamental units is called the absolute system. It is usually assumed that the units belonging to an absolute system are not subject to any variation, but the question very naturally arises as to whether we are justified in making this assumption. Comparatively recent studies in connection with the subject of radio-activity have revealed the fact that some substances emit particles and that as a consequence the masses of these bodies are very slowly diminishing. There is no evidence, however, that the kinds of material used in the construction of the standards of mass are radio-active, and there is every reason to believe that the standard kilogram has not changed in the 125 years since it was made. Likewise, we can find no reason for supposing that the unit of length would suffer any alteration. The time unit is constant if the average period of the earth's rotation is constant. If the earth were to contract, as it probably has during past ages, the speed of its rotation would be increased. On the other hand, the energy of tidal motion is derived from the energy of rotation of the earth; consequently some of this energy is converted into heat. If its energy of rotation is decreasing and there is nothing to compensate for this effect, the speed of rotation must be decreasing also. It has been estimated that the change in our time unit due to this cause might amount to one part in a million in the course of 8,000 years. It is therefore apparent that the change in the fundamental standards in any limited period of years is quite insignificant.

The system of units based upon the cm, the gram, and the second, is, for convenience, called the C.G.S. system; and the English system, based upon the foot, the pound, and the second as fundamental units, is called the F.P.S. system.

7. Types of Motion.—Motion of a point implies that the point is to be found at different places in a continuous path at different times, and that it passes successively through all points of this path as time progresses. Thus the two fundamental concepts related to the science of motion are length and time. If a body moves so that every line of the body remains parallel to its initial position, it is said to suffer translation. In this kind of motion the points of a body all trace equal paths and so, if the motion of one point is described, the motions of all points are known.

When a body moves so that two of its points are stationary, all the points lying in the line joining these two points will also be stationary and all other points of the body will describe circular paths about this line as an axis. Such motion is called rotation. In this case it is evident that some of the lines of the body do not remain parallel to their initial positions.

In some cases of rotation, the axis about which the rotation takes place is itself continually changing in position. A familiar case of this kind is that of a rolling body. Suppose the body is a cylinder and that it is rolling on a horizontal surface. The only line of the body that is stationary at a given instant is the line in its cylindrical surface that is in contact with the plane. This line is the instantaneous axis about which the rotation is taking place. A moment later some other line becomes the axis of rotation. We may study all the properties of this motion by treating it as rotation about the instantaneous axis, or, if we prefer, we may approach the matter in a different way. The rotation may be resolved into two motions, one of translation of the center of mass in a line parallel to the plane and the other of rotation about an axis through the center of mass parallel to the element of the surface. These two methods of attacking a given problem arrive at precisely the same result.

8. Vectors and Scalars.—When a point moves in a rectilinear path, its motion is completely described only if both the magnitude and the direction of the velocity are specified. The distance of a point from some former position that the point has occupied is called the displacement of the point from the former position. This is also a quantity having magnitude and direction. Quantities which, like velocity, displacement, and force, have both magnitude and direction, are called vector quantities;

while quantities, such as mass, volume, density, work, and heat, having magnitude only, are called scalars.

A vector may be represented by a segment of a straight line, whose direction is that of the quantity and whose length is chosen to represent the magnitude of the quantity. If necessary, an arrow is used to indicate the positive direction along the line. In representing the magnitude of a vector quantity by the length of a line, any scale may be used that is convenient; for example, twenty miles per hour might be represented by a line 20 cm long, or if this is too long to be convenient, it might be represented by a line 10 cm long. One-half cm then represents the unit of velocity, and all lengths of the figure should be interpreted in terms of this unit.

9. Addition of Vectors.—Scalars are added algebraically, but vector quantities must be added geometrically. This becomes evident as soon as we consider specific cases. If a sailboat has a velocity across a river due to the wind and another velocity down the river due to the current, its velocity with reference to the banks of the river is considered as the sum of the two velocities, and is found by the process known as vector addition. In Fig. 2, let *ab* be the velocity due to the wind, and *ac* the velocity due to the current; then *ad* represents the resultant velocity due to both and is called the vector sum of *ab* and *ac*. Since in this case the two velocities are assumed to be at right angles, the sum equals the square root of the sum of the squares of the two components.

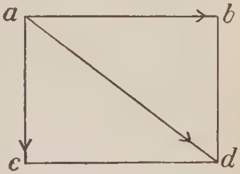


FIG. 2

10. Resolution of Vectors.—Suppose a sailing vessel starts from Chicago to go up Lake Michigan but, because of head winds, it is found necessary to tack across the lake. The master of the vessel is more interested in knowing how rapidly he is going northward rather than in his actual speed. Suppose that his observations show that he is sailing 12 miles an hour in a direction 32° north of east; can he find out how rapidly he is moving northward and how rapidly he is moving eastward? By constructing a parallelogram, as in Fig. 3, the vector representing 12 miles per hour may be resolved into two components, ab and ac.

$$ab = 12 \cos 32° \text{ and } ac = 12 \sin 32°$$

The sine of 32° = .5299, hence the boat is moving northward at the rate of 6.36 miles per hour and eastward at the rate of 10.8 miles per hour. This illustrates what may be done with any vector. It is not necessary that the components be at right angles, as in the case just considered; it is only necessary that the directions of the two axes along which the vector is to be resolved be known. The process then consists in completing a parallelogram whose diagonal is the given vector and whose sides are parallel to the given axes.

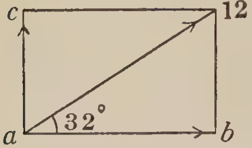


FIG. 3

11. Addition of Vectors not at Right Angles.—In Fig. 4, a and b are two vectors not at right angles, and θ is the angle between their directions. The sum of the vectors is the closing line of the triangle whose other sides are a and b. We shall resolve these vectors along two rectangular axes and for convenience choose the x axis in line with the vector a. We may resolve the other vector into two components c and d and make use of c and d instead of b; then the addition involves only the case discussed in § 9. Making this substitution,

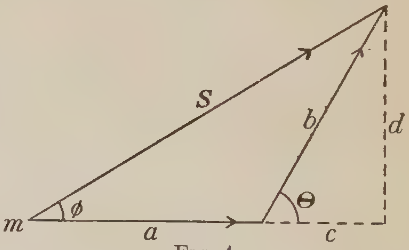


FIG. 4

$$S = a + c + d,$$

considered as a vector sum. But $c = b \cos \theta$ and $d = b \sin \theta$, hence S is the vector sum of a, $b \cos \theta$, and $b \sin \theta$. Since a and $b \cos \theta$ are in the same straight line, they are added algebraically; but $b \sin \theta$ is at right angles to a, hence the numerical value of the sum is

$$\begin{aligned} S &= \sqrt{(a + b\cos\Theta)^2 + (b\sin\Theta)^2} \\ &= \sqrt{a^2 + 2ab\cos\Theta + b^2\cos^2\Theta + b^2\sin^2\Theta} \\ &= \sqrt{a^2 + b^2 + 2ab\cos\Theta}. \end{aligned}$$

When the magnitude of S is found, its direction is determined by finding some function of the angle ϕ, as

$$\sin\phi = \frac{d}{S} = \frac{b\sin\Theta}{S} \text{ or } \tan\phi = \frac{d}{a + b\cos\Theta}.$$

12. The Sum of Several Vectors.—Any number of vectors lying in the same plane may be added in a manner similar to that used for two. Choose any two rectangular axes and resolve each vector into components along these axes. Let p be the magnitude of one of the vectors and Θ the angle it makes with the x axis. Calling the two components of p, x and y respectively,

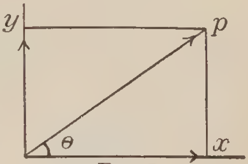


FIG. 5

$$x = p\cos\Theta \quad \text{and} \quad y = p\sin\Theta.$$

Each of the several vectors will have components along these two axes expressed by similar terms. All the x components may be added algebraically since they are in the same straight line, and similarly all the y components may be added.

$$\begin{aligned} X &= \Sigma x = \Sigma p\cos\Theta \\ Y &= \Sigma y = \Sigma p\sin\Theta \\ \text{and} \qquad S &= \sqrt{X^2 + Y^2}. \end{aligned}$$

Again letting ϕ represent the angle between S and the x axis, $\tan\phi = Y/X$; thus both the magnitude and direction of S are determined.

When several vector quantities are to be added, it is frequently found easier to use the graphical method. A convenient length is chosen to represent the unit in terms of which the vector quantity is measured. A vector is then drawn to represent each of the quantities to be added. The second vector begins where the first ends, and the third begins where the second ends, etc. The extreme ends of the diagram are connected by the line *s*, as shown in Fig. 6. The length of *s*, when expressed in terms of the unit used, represents the sum of all the vectors. Its direction is from *a* toward *e*. If the drawing is made with care, the result thus obtained is sufficiently accurate for most problems. Frequently the design of complicated structures is carried on by means of graphical methods. For example, the force in every piece of iron in a large bridge may be determined graphically if the load is known.

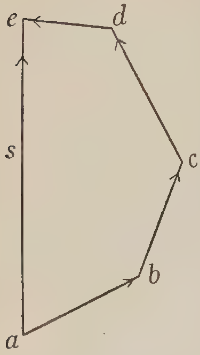


Fig. 6

PROBLEMS

1. A sidereal day is the interval of time between two successive passages of a fixed star across a meridian. Define a solar day. Draw a figure and show why one is longer than the other. Explain why solar days differ in length.

2. From a train running 60 ft. per second a body is thrown in a direction at right angles to the length of the train with a speed of 80 ft. per second. Find the velocity of the body with reference to the ground. Determine both its magnitude and its direction.

3. A point is moving with a velocity of 3 ft. per second along the diagonal of a square which is itself moving in the direction of one of its edges at the rate of 4 ft. per second. Find the actual velocity of the point. Solve this problem by a graphical method.

4. Solve problem 3 by a trigonometric method and compare the result with that obtained by the graphical method.

5. Rain that is falling vertically is observed by a person riding in an automobile. To him the paths of the drops appear to be inclined 30° with the vertical. The automobile is running 15 miles per hour (22 ft. per second). Find the velocity of the rain drops.

6. A steam boat is running south with a speed of 18 miles per hour, while the wind is blowing from the west with a velocity of 10 miles per hour. Find the angle made by the line of smoke from the smoke stack with the direction the boat is moving. Make a diagram.

7. If one of two forces is fixed in position and the direction of the other is variable, their resultant will depend upon the direction of the latter. If the maximum resultant that they can have is *R* and the

minimum resultant is r, find their resultant when they are at right angles.

If necessary, the student should consult §13 before solving problems 8, 9, and 10.

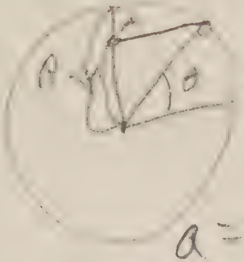
8. A rotating body has a uniform angular speed of 5 radians per second. How many revolutions will it make in 8 seconds?

9. A rotating body makes 10 revolutions in 2 seconds; what is its angular speed expressed in radians per second? $\theta = \omega t$

10. The velocity of the extremity of the minute hand of a clock is 16 times that of the hour hand, which is 3 inches long. Find the length of the minute hand.

R = amplitude

$V = \frac{2\pi R}{T} \cos$



$a = \frac{-4\pi^2}{T^2}$

θ radians

ω velocity r per sec

a acc. r per sec²

$a = -\frac{4\pi^2 R \sin}{T^2}$

$T = 2\pi\sqrt{\frac{x}{a}}$

CHAPTER II

FORCE AND MOTION

13. Uniform Velocity.—The velocity of a point in a straight line is the time rate of its change of distance from some fixed point in this line, *i.e.*, the rate of change of its displacement from this point. When the velocity is uniform it is numerically equal to the distance the point travels in any unit of time, or to the total distance divided by the time required to cover that distance. If s be the distance traversed in time t,

$$v = \frac{s}{t} \quad \text{or} \quad s = vt.$$

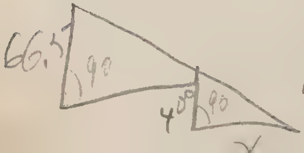
If a body has a uniform rotation about a fixed axis, any line of the body intersecting the axis at right angles describes an angle which increases uniformly with the time. The time rate at which the angle is described is the angular velocity and is usually represented by the letter omega (ω). Angular velocity may be expressed in degrees per second or in radians per second. If the angle θ is described in t seconds,

$$\theta = \omega t.$$

The radian is an angle whose intercepted arc is equal to the radius of the arc. Every time, therefore, that the radius can be laid off along a given arc, an angle equal to one radian has been measured off at the center. The number of times the radius can be laid off along an arc of length s equals s/r and the number of radians in the angle is s/r. Then if θ is an angle and s its intercepted arc,

$$\theta = s/r \text{ radians.}$$

If the angle θ were described in t seconds by the uniform



$$\frac{x}{400} = \frac{56.4}{66.5}$$

$x = 340g$

10.6 7

rotation of a radius, we might divide both members of the last equation by t and obtain two expressions for angular velocity,

$$\frac{\theta}{t} = \frac{s}{rt}.$$

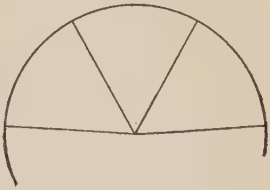


FIG. 7

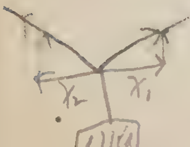
But s/t is the linear velocity of the point describing the arc; hence

$$\frac{\theta}{t} = \omega = \frac{v}{r} \quad \text{and} \quad v = \omega r.$$

This means that the linear velocity of the point describing the arc is equal to the angular velocity of the radius describing the angle multiplied by the length of the radius.

14. Acceleration.—When a body moves with a variable velocity the motion is called accelerated motion, and the time rate at which the velocity changes is called its acceleration. The simplest case of accelerated motion occurs when the motion is in a straight line and the acceleration is constant. This means that the speed is changing all the time at the same rate. For example, suppose a car has at a given instant a speed of three feet per second, and the speed increases uniformly and at the end of one second is 5 feet per second, at the end of two seconds it is 7 feet per second, at the end of three seconds it is 9 feet per second, etc. The change in speed is 2 feet per second in each second, hence the acceleration is 2 feet per second per second. 120 feet per second per minute would mean the same thing. When speed is decreasing the motion is still called accelerated motion, but the acceleration and velocity are oppositely directed.

The magnitude of the velocity, in the case of accelerated motion, is still defined as the time rate of motion; but we can no longer say that the speed is the distance traversed in a unit of time, since the speed is changing continually. The distance traversed in the unit time is the average speed during that unit, but it is neither the speed at the beginning of the time nor that



$y_1 + y_2 = $ [illegible]

$T_1 \cos\theta = T_2 \cos\theta$

T_1 [illegible]

at the end of it. If, in the expression, $v = s/t$, we make s very small, then t will be correspondingly small, and the average velocity during the time t will differ very little from either the velocity at the beginning or that at the end of the time t. This ratio, s/t, approaches a definite limit as the quantities s and t are made to approach zero. The limiting value of this ratio is written ds/dt, and is the expression for instantaneous velocity. Suppose a railroad train travels between two stations, a distance of s feet, in t seconds. The ratio s/t is the average speed during the t seconds, but it gives no idea of the speed at any particular instant. Suppose we wish to know the speed at the instant the locomotive starts to cross a particular bridge. If we could measure the time necessary to travel the length of one rail of the track and divide the length of the rail by this time, we should have the average speed of the locomotive while traversing this rail. Unless the speed is changing with great rapidity, this would be a fairly accurate measure of the instantaneous speed which we wish to know. For greater accuracy it is only necessary to shorten the space and the time—in other words, to approach the limiting case. It is this value of ds/dt that should be indicated at every instant by the speedometer of an automobile.

15. Force.—Before proceeding further with the study of motion, we must inquire into the conditions under which the motion of a body is produced or changed. The fact that a change of velocity of a body may be brought about by muscular effort is perfectly familiar. Anything which produces the same result as our muscular effort is called force; *i.e.*, we define force as that which may change the velocity of a body or, in other words, give it an acceleration. Accordingly we agree to measure force by the acceleration it can give a unit mass; and the defining equation for force is

$$F = Ma.$$

The branch of physics which deals with forces and their effects is called *dynamics*.

16. Units of Force.—If, in the equation $F = Ma$, we make $M = 1$, and $a = 1$, their product equals one and $F = 1$; *i.e.*, a unit of force is the force which, acting upon unit mass, gives it unit acceleration.

In the metric system the C.G.S. unit of force is the dyne. A *dyne* is a force which, acting upon a mass of one gram, will give it an acceleration of one cm per second per second.

In the English system, the F.P.S. unit of force is called the poundal. A *poundal* is a force which, acting upon a mass of one pound, will give it an acceleration of one ft. per second per second.

17. Gravitational Units of Force.—It is frequently found convenient to use as a force unit the earth's gravitational attraction for a unit mass and to call it by the same name as is used to designate the mass unit. A *gram force* is a force equivalent to the earth's attraction for a gram mass. Experiment shows that the acceleration due to gravity is about 980 cm per second per second. This means that if a gram force acts upon a gram mass the acceleration will be 980 cm per second per second; whereas if a dyne acts upon a gram mass the acceleration will be one cm per second per second. From this it follows that one gram force equals 980 dynes.

A *pound force* is a force equivalent to the earth's attraction for a pound mass. The acceleration due to gravity is approximately 32 feet per second per second, *i.e.*, the pound force will give a pound mass an acceleration of 32 feet per second per second, while a poundal will give it an acceleration of only one foot per second per second; hence a pound force equals approximately 32 poundals.

18. Conditions Necessary for Uniform Motion.—We have seen that, when a force acts upon a body of mass M, the body experiences a change of velocity and has an acceleration equal to F/M. When we know that a force is acting upon a body and the body has no acceleration, it may seem that we have a contradiction of this statement. For example, if a railroad train has a uniform speed on a horizontal track, we are certain that the locomotive is exerting force upon the train. However, we are equally certain that there is a frictional resistance to the motion of the train. The fact that these two opposing forces are equal explains why there is no acceleration. The resultant of all the forces acting upon the train is zero, and it is the resultant force that equals mass times acceleration. The condition under which uniform translational velocity is maintained is that the resultant of all the forces acting upon the moving body shall be zero.

19. Problem.—To illustrate the foregoing topics on force, let us solve the following problem. A car has a mass of 40,000 pounds. The frictional resistance to its motion amounts to a force of 300 pounds. Find the force which in 10 seconds will give it a speed of 8 feet per second on a horizontal track.

The acceleration is the time rate of change of velocity, *i.e.*, .8 feet per second per second, assuming it to be uniform.

$F = Ma = 40{,}000 \times .8 = 32{,}000$ poundals, needed to produce the acceleration. The total force required is the sum of this force and the force needed to overcome frictional resistance. However, before we can add these two forces, they must be expressed in terms of the same unit.

One poundal = 1/32 of a pound force, 32,000 poundals = 32,000 × 1/32 pound = 1,000 pounds force.

1,000 pounds + 300 pounds = 1,300 pounds, the force necessary to move the car with an acceleration of .8 feet per second per second.

20. Composition and Resolution of Forces.—Since force has magnitude and direction, it has the properties of a vector quantity. It has, in addition to these properties, a definite point of application and must therefore pass through this point. Another equal and parallel force not passing through this point of application would in general produce a very different result. A force, then, may be considered as a localized vector quantity, and it is represented by a vector having a definite position in space.

Two intersecting forces may be combined by the process already described in §11. The vector sum of the two forces is called their resultant. It is evidently the single force which, acting alone, will produce the same result as the two given forces.

The resultant of any number of forces, all passing through the same point, may be found by the method described in §12. The figure is called the polygon of forces, and the closing line of the polygon represents the resultant.

A single force may be resolved into two components in any two directions in the manner explained in § 10. Let *ab* in Fig. 8 represent a force. If we wish to find its components in the two directions *ac* and *ad*, we draw from *b* a line parallel to *ac* and another from *b* parallel to *ad*. Thus there is formed a parallelogram whose sides *ae* and *af* are the two components sought.

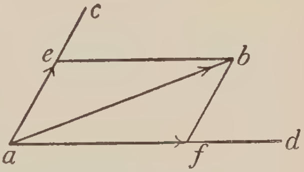


FIG. 8

21. The Law of Gravitation.—"Every particle of matter in the Universe attracts every other particle of matter with a force that varies as the product of the masses and inversely as the square of the distance between them." The statement of this law was first made by Sir Isaac Newton, and it is frequently referred to as Newton's Law of Universal Gravitation. When we are dealing with the case of the forces between given masses at different distances, *i.e.*, when the distance is the only variable, the force varies inversely as the square of the distance between the mass particles. This latter statement is called the law of inverse squares. A homogeneous sphere exerts a gravitational force upon any mass particle. This force varies inversely as the square of the distance between the center of the sphere and the mass particle. If the earth is regarded as a homogeneous sphere, its gravitational force upon bodies outside its surface may be considered as varying inversely as the square of the distance from its center; hence the acceleration due to gravity is different for different altitudes above the sea level.

Since the earth is not a sphere and the poles are nearer to the center than is the equator, it will be apparent that the acceleration will vary with the latitude, becoming greater at higher latitudes. Thus we see that the attraction of the earth for a body, *i.e.*, its weight, depends upon where the body is, and does not remain constant if the body is moved from place to place. The operation of centrifugal force due to the rotation of the earth is responsible for a further variation of weight with latitude. It follows, therefore, that while we may regard the absolute units as fixed in magnitude, the gravitational units are not.

22. Newton's Laws of Motion.—Newton formulated three propositions which are known as Newton's Laws of Motion and which are the axioms upon which the science of dynamics is based. The word momentum, as used in the second law, means the product of the mass of a body and the speed with which it moves, *i.e.*, momentum = Mv.

The First Law.—"Every body continues in its state of rest or uniform motion in a straight line except in so far as it is compelled to change that state by force impressed upon it." We have defined force as that which changes the motion of a body. This law merely states that a change of motion never occurs except as the result of force. If a body is at rest it will remain at rest unless some force acts upon it. If it is in motion it will continue in motion with uniform velocity unless some force acts upon it. We attribute any change of motion of a body to a force acting upon it. The change of velocity may be either a change of magnitude or a change of direction. The fact that the earth moves in an elliptical orbit rather than in a straight line is the most obvious proof that it is attracted by the sun.

impulse

$Ft = mv$

The Second Law.—"Change of momentum is proportional to the impressed force and takes place in the direction in which the force acts." In fact, change of momentum is *determined*

$F \propto mv - mv_0 \quad F = \frac{k\,m(v - v_0)}{t} = kma$

1 dyne

$Ft = k(mv - mv_0) \quad F = k\,1\,gm\,1\frac{cm}{sec^2}$

$W = mg$

$F = K\frac{mm_1}{r^2}$

by the impressed force and the time during which the force acts. One may double the change of momentum produced either by doubling the force acting for a given time or by doubling the time during which the force acts. If the unit of force is properly chosen, one may state this in the form of an equation:

Change of momentum $= Mv_1 - Mv_o = Ft$ from which

$$F = \frac{M(v_1 - v_o)}{t} = Ma.$$

$v_1 - v_o$ represents the change of velocity occurring in t seconds. In accordance with this equation, force is measured by the time rate of change of momentum which it produces. Since change of velocity per unit time is acceleration, it is also expressed as mass times acceleration. Thus it is seen that Newton's second law may furnish the basis for the method of measuring force which was first developed in §15.

This law implies further that a force will produce a change of momentum proportional to itself and in its own direction, whether the body is at rest or in motion and whether or not other forces are acting upon it. For example, if a bullet were shot horizontally over a body of still water, it would strike the water at the same instant as if it were dropped vertically. The force of gravity produces the same change of velocity in the vertical direction, irrespective of any horizontal velocity the body may have.

The Third Law.—"To every action there is an equal and contrary reaction or the mutual actions of two bodies are equal and opposite." This law points out that all forces are essentially "double ended." A simple example is the gravitational force between the earth and the moon. From the point of view of the moon, it might be said that the earth was attracted toward it, and from our point of view, the moon is attracted toward the earth. The fact is that the attraction between them is mutual. If a body is free to fall, the same force is operating

$V = \frac{S_2 - S_1}{t_2 - t_1} = \frac{s}{t}$

$a = \frac{v - v_1}{}$ [illegible]

on the earth and on the falling body. This force acts upon both for the same time and they will meet with equal momenta. All forces have this dual nature.

23. Uniformly Accelerated Motion.—Let v_0 equal the initial velocity of a body, and v equal the velocity t seconds later. During these t seconds let a uniform force act upon the body in the direction of its initial velocity; then the body will have a uniform acceleration in the direction of v_0. Since acceleration is the rate of change of velocity, the value of this acceleration will be

$$a = \frac{v - v_0}{t}.$$

Clearing this equation of fractions and solving for v,

$$v = v_0 + at.$$

This is equivalent to the statement that the final velocity is equal to the initial velocity plus the change of velocity. In a given problem, a may be negative; then v is decreasing as t increases. It will ultimately become zero and then negative, *i.e.*, the body will stop and then start in the reverse direction.

When the acceleration is constant, the average velocity for a given period may be found by averaging the initial and the final velocities. Let v' be the average velocity, then

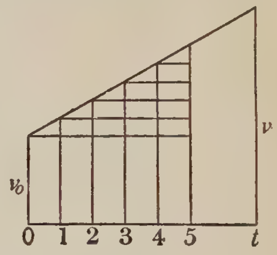


FIG. 9

$$v' = \frac{v_0 + v}{2}.$$

To show that this is the correct expression for the average velocity, plot the velocity as ordinate and time as abscissa, as in Fig. 9. When $t = 0$, the velocity is v_0; hence the first ordinate is v_0. When $t = 1$, the velocity is $v_0 + a$; when $t = 2$, the velocity is $v_0 + 2a$; when $t = 3$, the velocity is $v_0 + 3a$, etc. Since the acceleration is constant, a straight line may be drawn through the upper extremities of these ordinates. The average velocity is the average altitude of this figure, which is the height of the ordinate midway between the first and the last; and this is easily seen to be half the sum of the first and the last ordinates. If the acceleration were not constant, the average velocity could not be expressed in this way.

If we substitute for v its value, $v_o + at$,

$$v' = \frac{v_o + v_o + at}{2} = v_o + \tfrac{1}{2}\,at.$$

If v_o and a are in the same direction, the average velocity times the time t gives the distance that the body travels in time t. Calling this distance s, we have

$$s = (v_o + \tfrac{1}{2}\,at)t = v_ot + \tfrac{1}{2}\,at^2.$$

If v_o and a are oppositely directed, s must be interpreted as meaning the distance of the body from its initial position after t seconds. This case is easily explained in the discussion of falling bodies, and will be mentioned in §28.

The two equations, $a = \dfrac{v - v_o}{t}$ and $s = \dfrac{(v + v_o)t}{2}$, are simultaneous equations, and by multiplying them member by member, we may eliminate t and obtain an expression for v not involving t.

$$as = \frac{(v - v_o)}{t}\,\frac{(v + v_o)t}{2} = \frac{v^2 - v_o^2}{2}, \text{ and}$$

$$v^2 = v_o^2 + 2\,as.$$

24. Freely Falling Bodies.—While the force of gravity varies inversely as the square of the distance from the center of the earth, its variations over small vertical distances are so slight that we may regard it as practically constant. A constant force produces a uniform acceleration; consequently a falling body has uniformly accelerated motion, and the laws of falling bodies are obtained at once by merely substituting g, the acceleration due to gravity, for a in the equations of uniformly accelerated motion.

$$v = v_o + gt$$
$$s = v_o t + \tfrac{1}{2} gt^2$$
$$v^2 = v_o^2 + 2\,gs.$$

The acceleration g is always vertically downward, but v_o may be either up or down.

When the body starts from rest, the initial velocity v_o equals zero, and these three equations become

$$v = gt$$
$$s = \tfrac{1}{2} gt^2$$
$$v^2 = 2\,gs.$$

25. Uniformly Accelerated Rotation.—Angular acceleration is defined as the time rate of change of angular velocity. Representing angular velocity by ω, angular acceleration by α, and the angle described by θ, we may derive the equations for uniformly accelerated rotation in a manner similar to the method of §23. The equations are as follows:

$$\omega = \omega_0 + \alpha t$$
$$\theta = \omega_0 t + \tfrac{1}{2} \alpha t^2$$
$$\omega^2 = \omega_0^2 + 2\alpha\theta.$$

26. Path and Range of a Projectile.—When the initial velocity v_o has some other direction than the vertical, the path of the moving body becomes a curve. If the resistance due to friction in the air is considered, the problem is complicated; but we may neglect this force in obtaining an approximate solution, in which case the path of the projectile may be shown to be a parabola. Let v_o in Fig. 10 be the initial velocity. We shall first find the vertical height to which the body will rise. The vertical component of v_o is $v_o \sin \theta$, θ being the angle that v_o makes with the horizontal. θ is called the angle of elevation. The vertical component of the motion of the projectile is merely the motion of a freely falling body; therefore the equations of § 24 may be applied directly and

$$v^2 = v_o^2 \sin^2 \theta + 2\,gs.$$

When the projectile reaches its highest point $v = 0$; therefore

$$v_o^2 \sin^2 \theta + 2\,gs = 0 \text{ and } s = -\frac{v_o^2 \sin^2 \theta}{2g}.$$

The negative sign indicates that s and g are oppositely directed. The time required to reach this highest point is given by the equation

$$v = 0 = v_o \sin \theta + gt, \text{ from which}$$

$$t = \frac{-v_o \sin \theta}{g}.$$

$V_o \sin \theta$ and g are opposite in sign, so that the fraction representing t is positive.

The range of a projectile is the distance from the starting point to the place at which it again reaches the same level, represented by AB in the figure. To find this range, we first observe that if t, in the equation above, represents the time for the projectile to reach its highest point, $2t$ must represent the total time of flight or the time for the projectile to go from A to B. Secondly we recall that we are neglecting friction, and hence there is no force to impede the horizontal motion of the body. In such a case the horizontal component of the velocity of the projectile is constant. This component is $v_o \cos \theta$, and since the time that the body moves with this velocity is $\frac{2v_o \sin \theta}{g}$, the distance traversed is

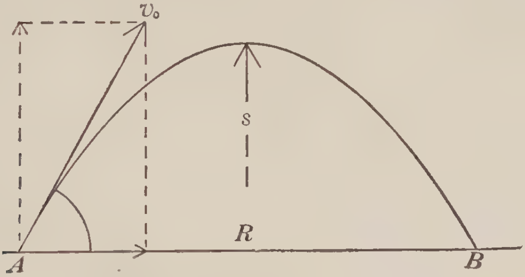


FIG. 10

$$R = \frac{2v_o^2 \sin \theta \cos \theta}{g}.$$

Thus it appears that if the initial velocity is known and the angle of elevation is given, the range may be computed.

The angle of elevation which gives the greatest range is easily found. $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$. If we substitute this value of $\sin\theta\cos\theta$ in the expression for the range, we have

$$R = \frac{v_0^2 \sin 2\theta}{g}.$$

R is a maximum when $\sin 2\theta$ is a maximum, *i.e.*, when $2\theta = 90°$ or $\theta = 45°$.

For long-range projectiles the velocities must be very high and consequently the friction in the air cannot be neglected. The projectile from an army rifle may lose half its initial velocity in the first two seconds of its flight. The equations that we have derived furnish only a first approximation, and large corrections must be made.

27. Distance Traversed in Any Second.—If the distance traversed in $t - 1$ seconds be subtracted from the distance traversed in t seconds, the result is the distance traversed in the last second. Call this s'; then, if the body starts from rest and has a uniform acceleration,

$$\begin{aligned} s' &= \tfrac{1}{2}at^2 - \tfrac{1}{2}a(t-1)^2 \\ &= \tfrac{1}{2}at^2 - \tfrac{1}{2}a(t^2 - 2t + 1) \\ &= \tfrac{1}{2}a(2t - 1). \end{aligned}$$

If t is any whole number of seconds, $2t$ is an even number and $2t - 1$ is an odd number; thus we see that the distances traversed in successive units of time vary as the series of odd numbers. When $t = 1$, $s' = \frac{1}{2}a$. When $t = 2$, $s' = \frac{3}{2}a$. When $t = 3$, $s' = \frac{5}{2}a$. When $t = 4$, $s' = \frac{7}{2}a$, etc.

28. Problem.—A body is projected vertically upward with a speed of 144 ft. per second. Where will it be and what will be its velocity at the end of 8 seconds?

Whether the upward direction or the downward direction is considered as positive is immaterial, provided one is consistent in his interpretation of data and results. In this case let us consider the upward direction as positive; then g equals -32 ft. per second per second, $t = 8$ seconds and $v_0 = 144$ ft. per second.

$v = 144 - 8 \times 32 = -112$ ft. per second, which means that the body is moving downward at the rate of 112 ft. per second.

$s = v_0 t + \frac{1}{2} gt^2 = 144 \times 8 - \frac{1}{2}(32 \times 64) = 1152 - 1024 = 128$ ft. This is positive, therefore the body is above its starting point by 128 feet. This 128 feet is evidently not the distance it has traveled in the 8 seconds.

To find how far the body rises before it stops and begins to fall, put v equal to zero, and we have the equation

$$0 = 144 - 32\,t, \text{ from which } t = 4.5 \text{ seconds.}$$

The distance traversed in these 4.5 seconds is

$$\begin{aligned} s &= 144 \times 4.5 - \tfrac{1}{2}(32 \times \overline{4.5}^{2}) \\ &= 648 - 324 = 324 \text{ feet.} \end{aligned}$$

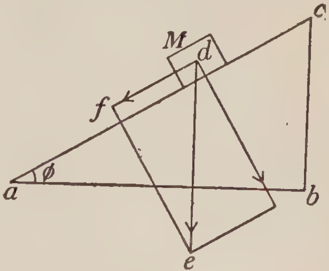


FIG. 11

29. Motion on an Inclined Plane.—Let a body of mass M be placed on an inclined plane which makes an angle ϕ with the horizontal, as shown in Fig. 11. The force of gravity acting upon the body is vertical and is equal to Mg. It may be represented by the vertical vector de. We resolve this force into two components, one parallel to the plane and one perpendicular to the plane. The component parallel to the plane is the one that produces the motion, and the one perpendicular to the plane does nothing but press the body against the plane. The triangles def and abc are similar; hence the angle def is equal to bac equal to ϕ. The force df that produces the motion equals $Mg \sin \phi$; assuming that there is no frictional resistance to motion, the force tending to move the body is constant and hence will produce a uniform acceleration. $f = Mg \sin \phi = Ma$; thus we see that the body will slide down the plane with an acceleration $g \sin \phi$ and all the laws of uniformly accelerated motion will apply to this case.

By making ϕ small, the acceleration may be made as small as desired, and the body may be made to move so slowly that its motion may be easily observed. This method of reducing the acceleration is sometimes used when an experimental verification of the laws is desired.

30. Other Cases of Diminished Acceleration.—In Fig. 12, let M be the mass of a car that is assumed to move on a smooth table without friction. A string passes over a very light frictionless pulley and is attached to a small body of mass m. The force of gravity on the mass m is mg dynes, and this equals the product of the total mass moved and the acceleration with which it is moved.

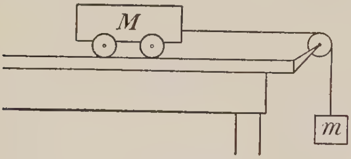


Fig. 12

$$mg = (M + m)a, \text{ from which } a = \frac{m}{M + m} g.$$

Under the conditions assumed, a is much smaller than g and may be made still smaller by reducing m. Since mg is the weight of the body, and a part of this gravitational force, equal to ma, is used in producing the motion, the remainder, $mg - ma$, must be the force applied to the string. By considering the part of the string where it is fastened to the car, it is evident that the expression Ma also represents the force in the string. That these two expressions for the same thing are consistent, is obvious from the equation above.

Suppose the same car to be placed on an inclined plane, as shown in Fig. 13. If m is sufficiently large so that the car is drawn up the plane, we have the resultant force

$$f = mg - Mg \sin \phi,$$

and this force is equal to the product of the total mass moved into its acceleration, *i.e.*,

$$mg - Mg \sin \phi = (M + m)a.$$

This acceleration may also be made as small as desired by making m and $M \sin \phi$ nearly equal. In this case the two expressions for the force in the string are $Mg \sin \phi + Ma$ and $m(g - a)$ respectively. The student should satisfy himself that these expressions are correct and that they are consistent with each other.

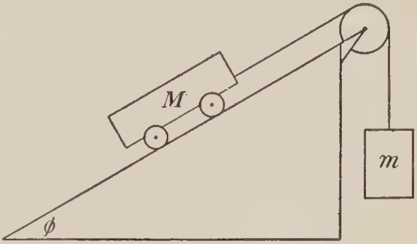


FIG. 13

31. Atwood's Machine.—Perhaps the best apparatus for the experimental study of the laws of uniformly accelerated motion is Atwood's machine. Two bodies having equal masses are suspended from opposite ends of a long string which passes over a light pulley that moves with very little friction, as illustrated in Fig. 14. The forces of gravity on the two bodies are balanced so that the system may be treated as if no force were acting upon it. A small rider of mass m is now placed upon one of the large bodies, and the system moves under the action of the gravitational force mg. The total mass moved by this force is $2M + m$, hence, representing the acceleration of the system by a, we have

$$mg = (2M + m)a \text{ and } a = \frac{m}{2M + m}g.$$

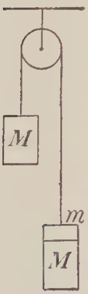


FIG. 14

Here, as in the other cases, two expressions for the force in the string may be written, either of which may be reduced to the other. This equation takes no account of the mass of the wheel, which usually cannot be ignored. The acceleration will actually be less than that computed from the equation, because the wheel must be set in motion as well as the two bodies. A scale for measuring distances traversed by the moving bodies and a pendulum for measuring time are essential parts of the apparatus.

32. Uniform Circular Motion.—When a point moves with uniform speed along the arc of a circle, its velocity is constantly changing in direction, and hence the point has an acceleration even though there is no change in the magnitude of its velocity.

Let a point move with uniform speed along the arc mn of the circle whose center is o in Fig. 15. When the point is at m, its velocity is v, and is perpendicular to the radius r. When the point has reached n, its velocity is v', perpendicular to the radius r'. v' is a vector which is obtained by adding the vector u to the vector v; hence u is the change in velocity. If the points m and n are taken very near together, v and v' are very nearly parallel and u is perpendicular to v, *i.e.*, the change of velocity is toward the center of the circle. If t units of time are required for the point to move from m to n, this change of velocity occurs in t units of time, and u/t is the average rate of change of velocity. In the limiting case under consideration, *i.e.*, when the distance between m and n approaches zero, this average rate of change of velocity becomes the acceleration and $a = u/t$. From the figure it is apparent that, for small angles $u/v = \Theta$ or $u = v\Theta$. Substituting this value of u in the equation $a = \frac{u}{t}$, we have $a = \frac{v\Theta}{t} = v\omega$, ω being the angular velocity of the radius. v/r may be substituted for ω, in which case

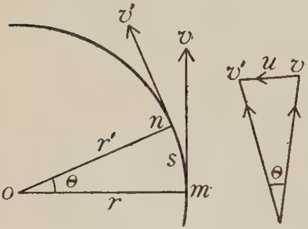


FIG. 15

$$a = \frac{v^2}{r}.$$

Substituting ωr for v in the same equation, we obtain an expression for a in terms of ω, thus:

$$a = \omega^2 r.$$

This is called the centripetal acceleration. It is always directed toward the center of the circle in whose circumference the point is moving. If v is measured in cm per second, u is also measured in cm per second, and u/t must be measured in cm per second per second. When the foot is taken as the unit of length, the acceleration a is measured in feet per second per second.

If a mass particle m has a uniform circular motion, it will have an acceleration toward the center, and hence a continuous force,

$$ma = \frac{mv^2}{r},$$

must act to pull it toward the center. This force is called the centripetal force; and its equal and opposite reaction, called the centrifugal force, is the force that the mass exerts away from the center. These forces vary as the mass, as the square of the speed, and, for a given speed, inversely as the radius of the curve.

If the mass is measured in pounds, the velocity in feet per second, and the radius in feet, the centripetal force is expressed in poundals. This is evident since ma measures force in absolute units.

Some knowledge of centrifugal force and its laws comes to nearly everybody from everyday experience. Everyone knows that an automobile is more likely to skid when rounding a curve at high speed than when running at low speed. Furthermore, it is more likely to skid if the radius of its path is short than if it is long. When flywheels break, it is because at high speeds the centrifugal force becomes very great. The earth is flattened at the poles because of its rotation.

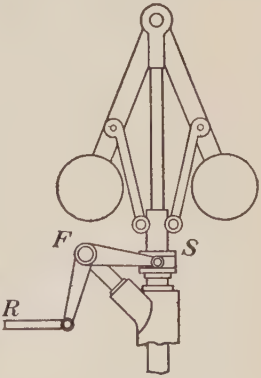


FIG. 16

The facts regarding centrifugal force find numerous application in such devices as Watt's governor for controlling the speed of a steam engine, the cream separator, the centrifugal drying machine, and the centrifugal pump. Fig. 16 illustrates the Watt governor. Two heavy balls are fixed at the ends of two rods and the rods are hinged together at the top. The frame is made to rotate as the engine runs, being geared to some rotating part. If the speed increases the balls move outward and the sleeve S is lifted. This moves the arm of the right-angled lever whose fulcrum is at F, and the rod R is drawn toward the right. This rod is connected to the valve gear of the engine in such a way that the steam supply is reduced when the rod moves to the right. When the speed is decreased by an increased load on the engine, the balls fall slightly, the rod R is moved toward the left and more steam is admitted to the cylinders of the engine. In this way the speed is kept nearly constant as the load varies.

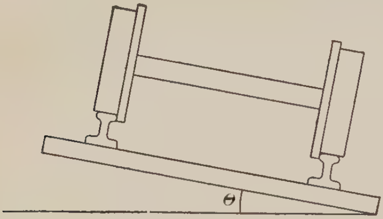


FIG. 17

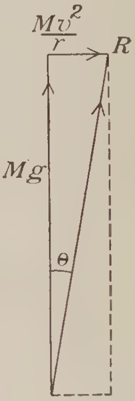


FIG. 18

The outer rail of a railroad curve is made higher than the inner one in order that the reaction of the rails against the wheels of the car may have a horizontal component to balance the centrifugal force when the train is running around the curve. Let Fig. 17 represent the track and the wheels of the car. The line running across the upper surfaces of the rails at right angles to their length makes an angle θ with the horizontal. The problem is to find how large this angle should be for a given radius of curvature and for a given speed of the train. In Fig. 18, let R be the reaction of the rails against the car. This should be perpendicular to the line joining the rails. R must have two components, one vertically upward to balance the force of gravity equal to Mg, the other horizontal to balance the centrifugal force and equal to $\frac{Mv^2}{r}\cdot$ From the figure it is seen that $\tan \theta = \frac{Mv^2}{r} \div Mg$, from which

$$\tan \theta = \frac{v^2}{gr}.$$

By computation of this kind the engineer is able to make a design that will meet the needs for that speed most likely to be used on a given curve.

33. Hooke's Law.—If a wire is stretched by a force, and careful measurements are made of the change of its length, it is found that the change of length is proportional to the applied force if the distortion is not too great. Likewise, if any elastic body suffers distortion within certain limits, it will be found that the amount of the distortion is proportional to the force causing it. This relationship was first announced by Robert Hooke, and is known as Hooke's Law. We shall have occasion to refer to this law again when studying the subject of elasticity; but for the purpose of developing the next paragraph we need only the simple facts here stated. When a coiled spring is stretched, each portion of the wire is twisted and the forces of reaction are proportional to the distortion. By varying the load on a spring and measuring the stretch produced by each load, it is easy to show that the distortion of the spring as a whole and the reacting force which it exerts are proportional.

34. Simple Harmonic Motion.—Every branch of physics is concerned with vibratory motions. In nature vibratory motions are found everywhere; the branches of a tree after being disturbed by the wind execute vibrations, a body floating on the surface of water rises and falls with the waves, bodies that emit sound vibrate and communicate their vibrations to the air. As a simple illustration of a vibratory motion, consider the motion of a body suspended upon a coiled spring. If the body is pulled downward and then released, it will vibrate in a vertical path, and some of the characteristics of its motion are very apparent. It has a maximum velocity near the middle of its path and comes to rest at each end of it. When the body is moving toward the middle position its velocity is increasing, and when it is moving away from this position its velocity is decreasing. This is equivalent to saying that the acceleration is always toward the middle point of the path. As illustrated in Fig. 19, it is convenient to fasten an index to the vibrating body and to place a scale parallel to its direction of motion so that the index is at the zero of the scale when the body is at rest. When the body is vibrating and is above the position of rest, it is considered as having a positive displacement, and the distance of the index above the zero of the scale is a measure of this displacement. When the stretch of the spring is increased and the index is below the zero of the scale, the displacement is considered negative. If the body is moving upward its velocity is considered positive. At the instant the moving body passes through its middle position the force exerted by the spring and the gravitational force are equal and opposite. The resultant force is therefore zero. When the displacement is negative the force of the spring is greater than the weight of the body, and the resultant force, in accordance with Hooke's Law, is proportional to the displacement and opposite in sign. Since $F = Ma$, the acceleration is proportional to the force, which in turn is proportional to the displacement and opposite in sign. It follows that in this case the acceleration of the vibrating body is proportional to the displacement and opposite in sign. Whenever a body vibrates under the action of elastic forces, its motion has these same characteristics. Such a motion is called *simple harmonic motion.*

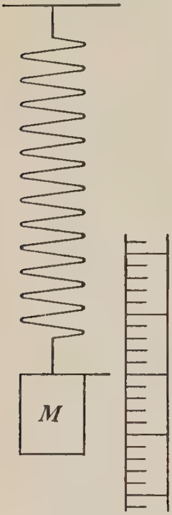


FIG. 19

The *phase* of simple harmonic motion is the fraction of a period that has elapsed since the vibrating body last passed the middle of its path in the positive direction.

The *period* of the motion is the time required for a complete vibration.

The *amplitude* is the distance from the position of equilibrium to one extremity of the path.

Having defined simple harmonic motion as a motion such that the acceleration is proportional to the displacement and opposite in sign, we may write the equation of the motion as follows:

$$a = -kY,$$

a being the acceleration, Y the displacement, and k a constant to be determined.

35. Relation of Simple Harmonic Motion to Circular Motion.—In Fig. 20 we shall assume that the point B has a uniform speed in the circumference of the circle and that P is its projection on a vertical diameter. As B moves around on the circumference, P will move up and down on the diameter, the amplitude of its vibration being the radius of the circle. Our immediate purpose is to determine the nature of the motion of the point P.

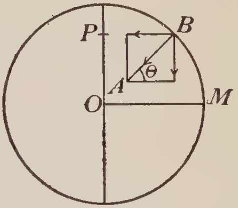


FIG. 20

The acceleration of a point having uniform circular motion has been found to be V^2/R and its direction is toward the center (§32). Since

$$V = \frac{2\pi R}{T},$$

$\theta = \omega t$

$$\frac{V^2}{R} = \frac{4\pi^2 R}{T^2}.$$

$v \frac{2\pi R}{T} \cos \omega t$

This is the acceleration of the point B in the figure. It may be represented by the vector A. The acceleration of P is the vertical component of A and is designated as a. From the figure it is evident that

$$a = A \sin \theta.$$

$\omega T = 2\pi$ [illegible]

Substituting for A its value, $\frac{4\pi^2R}{T^2}$,

we have

$$a = \frac{-4\pi^2R}{T^2}\sin\theta.$$

This is the value of the acceleration of P. In the first and second quadrants, $\sin\theta$ is positive, and in the third and fourth quadrants, it is negative. When $\sin\theta$ is positive, a is downward, and when it is negative, a is upward. Since it is customary to consider the upward direction as positive and the downward direction as negative, the value of the acceleration is given both in magnitude and direction by the above expression with its negative sign.

An inspection of the figure shows that $R\sin\theta = Y$. Substituting Y for $R\sin\theta$ in the equation for a, we have

$$a = \frac{-4\pi^2Y}{T^2} = -\omega^2Y.$$

Since ω is constant, this means that the projection on a diameter of a point having uniform motion on the circumference of a circle has a motion such that its acceleration is proportional to the displacement and oppositely directed. But this, by definition, is simple harmonic motion.

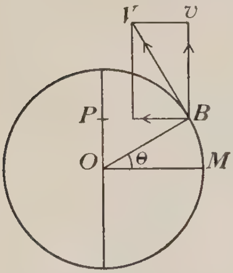


FIG. 21

Thus we have shown that simple harmonic motion, as defined at the outset, is the same as the motion of the projection on a diameter of a point having uniform speed in the circumference of the circle. Also we have shown the value of the constant k in the preceding paragraph to be ω^2.

36. Velocity of a Point Having Simple Harmonic Motion.—In Fig. 21 we assume P to have simple harmonic motion; and the period, or time of a complete vibration, is taken as T. The velocity of the point B is $\frac{2\pi R}{T}$. This velocity is represented by the vector V. The point P has a velocity v that is the vertical component of the velocity V. This is evident, since P moves vertically as fast as B does but has no horizontal motion. We therefore divide V into two components, one vertical and the other horizontal. The vertical component

$$v = V \cos \theta = \frac{2\pi R}{T} \cos \theta.$$

37. Time and Phase Relations.—The angle θ has now been found to vary at a uniform rate, *i.e.*, the radius OB has a uniform angular velocity; hence, if ω represents the angular velocity,

$$\theta = \omega t,$$

t being the time that has elapsed since the phase was zero. Substituting ωt for θ, the equations for displacement, velocity, and acceleration take the following forms:

$$Y = R \sin \omega t$$

$$v = \frac{2\pi R}{T} \cos \omega t$$

$$a = -\frac{4\pi^2 R}{T^2} \sin \omega t = -\omega^2 Y.$$

ωt is called the phase angle. When the phase angle is 30° the phase is evidently 1/12 of a period; when it is 45° the phase is 1/8 of a period; when ωt is 135° the phase is 3/8 of a period, etc. The terms phase and phase angle are frequently used interchangeably.

38. Problem.—The crosshead of an engine is the heavy piece of metal that is fastened to the piston rod. It slides back and forth as the flywheel revolves with a motion that is approximately simple harmonic. In the design of an engine it is necessary to know what forces will be used in causing the motion of this heavy piece of metal. Assume that the crosshead has a mass of 200 pounds and the length of stroke of the engine is 2.5 feet. The wheel makes 180 revolutions per minute.

We have proved that the acceleration $a = -\omega^2 Y$. Since $\omega = 2\pi n$ and $n = 3$ revolutions per second, $\omega = 6\pi$ radians per second and $\omega^2 = 36\pi^2 = 360$, calling π^2 equal to 10. The maximum value of Y is 1.25 feet; hence the maximum value of a is $A = 360 \times 1.25 = 450$ feet per second per second. Maximum force $= Ma = 200 \times 450 = 90{,}000$ poundals $= 2{,}812$ pounds. This is the force that must be acting upon the crosshead every time it reaches the end of the stroke. At other places in the path of motion the force is less, and at the middle of the stroke it is zero. It is evident that forces of this magnitude ought not to be ignored in designing machinery.

39. Problem.—In order that the reader may be sure that he understands how the velocity and acceleration of simple harmonic motion change with time, it is suggested he fill out the following table, first from observation of a figure representing a body having a simple harmonic motion, second from a study of the algebraic equations that we have derived. Use V and A to represent maximum values of velocity and acceleration respectively. Place a positive or a negative sign before each symbol.

Phase angle	Displacement	Velocity	Acceleration
0°		$+V$	
45°	$+.707\,R$		$-.707\,A$
90°			
135°			
180°			
225°			
270°			
315°			

40. Composition of Vibratory Motion with Rectilinear Motion.—If a vibrating body is given a uniform velocity at right angles to the direction of its vibration, any point upon it may be made to trace a curve from which it is possible to determine the nature of the vibration. If the vibratory motion is simple harmonic, the curve is called a sine curve and its equation is $y = Y \sin x$, x being proportional to ωt. The method of constructing a sine curve is illustrated in Fig. 22.

Construct a circle and draw two diameters at right angles for x and y axes. Divide each quadrant into four equal parts and through the points of division draw lines parallel to the x axis. The distance from one of these lines to the next represents the distance in the y direction that the moving point goes in time $T/16$. Starting with a point, as a, lay off on the x axis a series of equal segments, each of which is to represent the distance the point moves in the x direction in time $T/16$. If the point starts from a and has both these motions at the same time, it will trace the curve $a\ b\ c\ d$ etc.

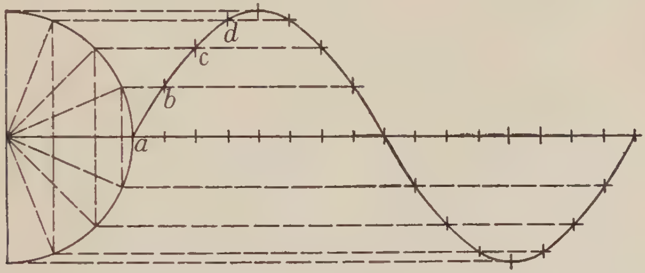


Fig. 22

A method making use of curves of this kind is developed in the study of alternating electric current phenomena.

PROBLEMS

1. A body having a mass of 2 lbs. rests upon a smooth horizontal plane. A horizontal force of 3 lbs. acts upon it for 4 seconds. What velocity is produced? F = Ma

2. If, in the preceding problem, the force were vertically upward, what acceleration would the body have?

3. A car with its load weighs 50,000 lbs. It runs up a 2% grade with an acceleration of .8 ft. per second per second. Frictional

resistance amounts to a force of 400 lbs. Find the force necessary to propel the car.

4. In the preceding problem, what force would have to be applied to the car to make it run down the grade with the same acceleration?

5. An elevator car weighing 350 lbs. is counterpoised by an equal weight on the other end of the cable which runs over a pulley whose mass may be neglected. A load of 1,000 lbs. is put into the car and it is then lifted for 5 seconds with an acceleration of 3 ft. per second per second. Then it moves for 4 seconds with a uniform speed, and finally with a uniform retardation such that it is brought to rest in 3 seconds more. Find the force that must be applied to the system in each case to produce these motions. Neglect friction.

6. A body whose mass is 3 kilograms slides on ice with an acceleration of 500 cm per second per second. The frictional resistance to its motion amounts to 1/16 of its weight. Find the force necessary to produce this motion.

7. How many dynes are necessary to lift a body whose mass is 3 kilograms vertically upward with an acceleration of 500 cm per second per second?

8. Two horizontal forces, 3 and 4 lbs. respectively, act at right angles upon a body having a mass of 20 lbs. What velocity will the body have after 3 seconds?

9. A dynamometer or draw scale is suspended in an elevator and a mass of 12 lbs. hung upon its hook. The elevator is then started with an acceleration of 3 feet per second per second, first upward then downward. What is the indication of the dynamometer in each case? If the experiment were repeated with a platform scale instead of the draw scale, what would be the result?

10. Two ropes are used in supporting a body weighing 200 lbs. One is horizontal and the other makes an angle of 30° with the vertical. Find the force in each rope.

11. A body weighing 100 lbs. at the earth's surface would have what weight 8,000 miles from the earth's surface? Assume the earth's radius to be 4,000 miles.

12. A body weighing 100 lbs. at the earth's surface would have to be carried how far from the earth to weigh only half as much?

13. A body starting from rest and having an acceleration of 20 cms per second per second will go how far in half a minute? What speed will it have at the end of the half minute?

14. A train has a speed of 60 ft. per second at a given instant and 12 seconds later its speed is 42 ft. per second. What is its acceleration, assuming it to be constant? When will the train come to rest? How far will it travel after the first instant mentioned?

15. A stone is thrown vertically upward with a velocity of 96 ft. per second. How high will it rise and when will it return to the ground?

16. A stone is thrown to the top of a flag pole and it reaches the ground after 5 seconds. How high is the pole?

17. A bullet is shot vertically upward with a speed of 160 ft. per second. Find the space passed over in 10 seconds. What is the meaning of s in the formula $s = v_0t + \frac{1}{2}at^2$?

18. Assume that a body could slide without friction on a plane inclined 30° to the horizontal. If the body is projected up the plane with a speed of 980 cms per second, how far will it go before coming to rest? How far will it go in 5 seconds, and where will it be then?

19. How far will a freely falling body move during the fifth second of its fall? (The acceleration due to gravity is approximately equal to 32 ft. per second per second.)

20. A stone thrown vertically upward remains in the air 8 seconds; how high does it rise?

21. A tower is 288 ft. high. At the same instant that one body is dropped from the top of the tower, another is projected vertically upward from the bottom and they meet half way. Find the initial velocity of the projected body and its velocity when it meets the falling body.

22. A bullet is shot upward with a velocity of 320 ft. per second, and 4 seconds later a second bullet is shot upward with a velocity of 190 ft. per second. Will they ever meet? Show how you know.

23. A stone is dropped from a high altitude, and 3 seconds later another is projected downward with a velocity of 150 ft. per second. When and where will the second overtake the first?

24. From a balloon which is ascending at the rate of 32 ft. per second a stone is dropped and it reaches the ground in 16 seconds. How high was the balloon when the stone was dropped?

25. An automobile moving with uniform speed of 20 miles per hour passes another which is standing still. The second one starts at the instant the first one passes and runs with uniform acceleration. What will be the speed of the second when it overtakes the first?

26. A train is running on a horizontal track at the rate of 30 miles per hour when the steam is suddenly shut off. How far will the train run before it stops if the resistance to its motion is equal to 16 lbs. force per ton of mass of the train?

27. A bullet with an initial velocity of 1,600 ft. per second strikes a target 3,600 ft. distant with a velocity of 1,000 ft per second. If the path of the bullet is assumed to be horizontal, compare the average resistance of the air to the weight of the bullet.

28. The length of an inclined plane is L and its height is H units. Show that if a body slides without friction down the plane, its velocity is the same at the bottom as if it had fallen freely through the height H.

29. If a body were thrown from the top of a high tower in a horizontal direction with a speed of 32 ft. per second, plot its path by finding its position at the end of each of the first four seconds. Use a scale of 64 ft. to the inch. What kind of curve is this? How do you know?

30. A rifle is fired while pointing at an angle of 45° with the horizontal. The bullet leaves the gun with a speed of 362 ft. per second. Find the range and the height to which the bullet rises. Suggestion: Resolve the initial velocity into vertical and horizontal components. Evidently each is 256 ft. per second. The horizontal motion is uniform and the vertical motion is uniformly accelerated, provided we neglect friction of the air.

31. A mass of 50 grams is placed on an inclined plane and a force of 30,000 dynes applied to it parallel to the plane. The mass is found to move up the plane with an acceleration of 100 cms per second per second. Find the angle that the plane makes with the horizontal. Friction is neglected.

32. Two bodies, 3 lbs. and 5 lbs. respectively, hang on opposite ends of a cord which passes over a frictionless pulley whose mass may be neglected. Find the acceleration with which the bodies will move and the force in the cord.

33. A mass of 24 lbs. lies on a table and is fastened by a cord which passes over a pulley at the edge of the table, to a mass of 8 lbs. suspended in the air. Frictional resistance to motion amounts to 96 poundals. What is the acceleration of the system when it is released, and what is the force in the cord?

34. A mass of 20 lbs. rests upon a smooth plane inclined 30° to the horizontal. A string fastened to it passes to the top of the plane and over a pulley, and from the end of the string is suspended a mass of 12 lbs. Find the acceleration of the system and the force in the string.

35. A body slides down a plane 100 ft. long, inclined at an angle of 45° with the horizontal. It then slides up a plane inclined at an angle of 30° with the horizontal, this latter motion being the result of the momentum gained in sliding down the first plane. How far does it go before coming to rest? Neglect friction.

36. A force of 3 lbs. acts vertically upward upon a mass of 2 lbs. for 5 seconds. How far does the mass move in the 5 seconds? What will the mass do if the force ceases to act at the end of the 5 seconds?

37. A mass of 25 grams slides down a plane inclined 45° to the vertical with an acceleration of 6 ft. per second per second. Find the force due to friction and express it in gravitational units. 13 gms.

38. A mass of 20 grams moves with uniform speed in a circular path of 50 cms radius and makes 90 revolutions per minute. Find the centripetal force and express it in gravitational units.

39. A car of 20 tons mass moves around a curve of radius 300 ft. at a speed of 12 ft. per second. Find the horizontal force on the rails. Express it in both absolute and gravitational units.

40. A bullet on the end of a thread 60 cms long describes a circular path in a vertical plane. What is the lowest speed that it may have at the highest point of its path and still maintain its circular motion?

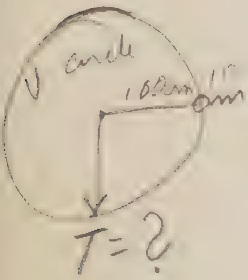
41. A small body moves in a horizontal circle having a radius of 3 ft. The centripetal force is 10 times the weight of the body. Find the angular velocity of the string connecting the body to the center.

42. A point describes simple harmonic motion of amplitude 100 cms and period of 8 seconds. Find its velocity and acceleration when its phase is 1/8 of a period; also when the phase is 3/8 of a period.

43. Find the displacement, the velocity and the acceleration of the point of the preceding problem when the phase angle is 120°.

44. If a mass of 100 grams had the motion described in problem 42, what would be the force acting upon it, (a) 1 second after it passed the middle of its path, (b) 2 seconds after passing the middle of its path, (c) 4 seconds after passing the middle of its path?

45. The crosshead of an engine weighs 160 lbs. It has a motion that is approximately simple harmonic with an amplitude of 1.5 ft. Find the maximum force used in moving it, if the engine is making 120 r.p.m.



CHAPTER III

WORK, ENERGY, AND POWER

41. Work.—The word work, as found in general use, has a somewhat indefinite meaning; but as used in a scientific sense, work is said to be done by a force when its point of application has a component of motion in the direction of the force. Two factors enter into the measurement of work, force and the distance through which its point of application is moved in the direction of the force. If there is no motion in the direction of the force, there is no work done by the force. The foundation of a house exerts force in supporting the house, but it does no work. Agents that do work are those like engines and motors that can produce motion while exerting force. The work done by a force is the product of the force and the distance its point of application moves in the direction of the force. The defining equation for work is

$$W = FS.$$

42. Units of Work.—Since $W = FS$, W may be made equal to unity by making each of the two factors, F and S, equal to unity. This signifies that a unit of work is done when a unit of force moves a body through a unit distance in the direction of the force.

In the C.G.S. system the unit of work is the erg. An *erg* is the work done by one dyne when its point of application moves one cm in the direction of the force.

In the F.P.S. system the unit of work is the foot-poundal. A *foot-poundal* is the work done by a force of one poundal when its point of application moves one foot in the direction of the force.

As a matter of convenience, several other work units have come into general use. The erg is so small that a multiple of it, called a joule, is frequently used. A joule equals 10^7 ergs.

When the force is measured in gravitational units, it is convenient to express the work done in gravitational units.

The foot-pound and the kilogram-meter are the gravitational work units most often used.

A foot-pound is the work done by a force of one pound when its point of application moves one foot in the direction of the force.

A kilogram-meter is the work done by a force of one kilogram when its point of application moves one meter in the direction of the force.

In computing the work done when a body is lifted vertically, it should be noted that the force used in lifting the body is equal to its weight only if the velocity is uniform, *i.e.*, if there is no acceleration.

43. Work Done When the Motion is not in the Direction of the Force.—A rope supporting a heavy weight may be performing a useful service, but the force transmitted through the rope does not do work unless there is motion produced. Likewise, if the rope and the weight it supports are moved horizontally, there may be work done, but it is not done by the force in the rope since this force can neither produce nor aid in producing a horizontal motion. In order that a force may do work, there must be some motion in the direction of the force. If a force, as in Fig. 23, acts in the direction indicated by the arrow, because of the operation of gravity the motion may be from O to A. This motion may be resolved into two components, OB in the direction of the force, and OC at right angles to its direction. The work done is $F \times OB$ which equals $F \times OA \cos \theta$. This amounts to the same thing as finding the component of the force in the direction of the motion and multiplying this component by the total distance moved.

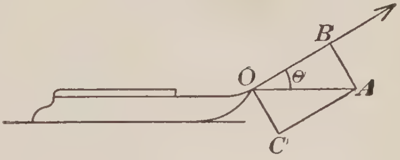


FIG. 23

A good illustration of these ideas is sometimes seen in a railroad yard. It is occasionally found convenient to move a car on a track by making use of a locomotive that happens to be on another track. A heavy pole is used to bridge the gap between the tracks; then the locomotive, pictured as *L* in Fig. 24, may push the car *C* along the other track. To find the work done in terms of the force *F* transmitted by the pole, we may resolve *F* into two components, one along the track and the other at right angles to the track. The one in the direction of the motion is *F* cos θ, and the work done by it is $D \times F \cos \theta$, *D* being the distance the car is moved.

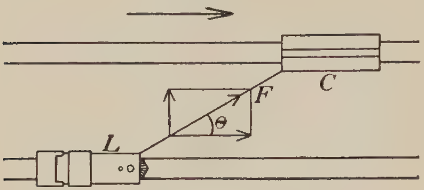


FIG. 24

44. Problem.—A ton of coal is lifted out of a mine 100 feet deep; how much work is done on the coal?

At the beginning of the operation the force is greater than the weight of the coal by an amount necessary to give it an acceleration. As the coal approaches the surface of the ground, its speed is retarded so that the force lifting it is less than its weight. The average force during the entire operation is equal to the weight of the coal, so that the work done is simply $2{,}000 \times 100 = 200{,}000$ foot-pounds.

45. Problem.—3,000 pounds of coal are drawn along a paved street for a distance of 1,000 feet. Find the work done.

The author of this problem is unable to solve it. Why?

46. Energy.—When work is done on a body, it frequently happens that, because of this fact, the body itself is capable of doing work. Work is done when a clock is wound and the weights lifted to the upper part of the case. As a result, the clock may run for a week. Work is done by the expanding gas due to the explosion in a gun when a projectile is fired, and as a result the projectile does considerable work when it is stopped. The clock weight at rest and the projectile in motion are both said to possess energy, because they are capable of doing work, and the measure of the energy is the work they can do. We define energy as the capability of doing work, and we measure it in work units.

47. Potential Energy.—Potential energy is the energy possessed by a system because of the relative position of its parts. The clock weight and the earth considered as a system possess energy because the position of the weight is such that the force of gravity may draw the weight toward the earth, thereby doing work. If a coiled spring is stretched, it possesses potential energy because the relative position of its parts is so altered that elastic forces are brought into action, and these forces will do work while the spring is contracting to its initial length. The measure of the potential energy of a system is the amount of work it can do while its parts are approaching that configuration in which they are incapable of doing more work. The energy associated with a clock weight is Mgh, Mg being the weight of the body and h the distance it can fall.

In the case of the stretched spring, the force varies from a maximum to zero in accordance with Hooke's Law, and the average force is half the maximum value. If F is the maximum force exerted by the spring and L is the distance through which the movable end of the spring is displaced, the potential energy stored in the spring is $FL/2$.

48. Kinetic Energy.—Kinetic energy is the energy a body possesses by virtue of its motion. The kinetic energy of a body may be determined by finding either how much work is done in setting it in motion or how much work it can do while being brought to rest. We shall use the former method and, for the sake of simplicity, make the following assumptions: that the body is at rest, that its mass is M, that it is acted upon by a force F, and that this force produces an acceleration equal to a. The measure of the force F is Ma. If the force acts for t seconds the body will acquire a velocity equal to at and in that time it will move a distance equal to $\frac{1}{2}\,at^2$. The work done by the force is

$$FS = Ma \times \tfrac{1}{2}\,at^2.$$

Substituting V for at, we have

$$\text{Work} = \frac{1}{2} MV^2.$$

This expression, representing as it does the work done in producing the velocity V, also represents the kinetic energy of the body. The result is expressed in absolute units. This is apparent if we consider that we substitute Ma for force, and Ma measures force in absolute units. If the mass is expressed in pounds, the force Ma is expressed in poundals and therefore the work is in foot-poundals.

49. Problem.—If a projectile having a mass of 10 pounds is fired vertically upward with a velocity of 640 feet per second, find its kinetic energy when it starts and its potential energy when it reaches its highest point. Disregarding friction, the two results should be equal.

50. Energy of Vibratory Motion.—If a pendulum consisting of a heavy piece of metal suspended on a long cord is drawn aside, it is at the same time raised to a higher level and therefore the system consisting of the pendulum and the earth has received a certain amount of potential energy. When the pendulum is released it gains velocity and therefore acquires kinetic energy. When it reaches its lowest point, the potential energy transferred to the system when the pendulum was raised has all been converted into kinetic energy. As it passes the middle of its path it begins to lose its kinetic energy and the system gains in potential energy until, at the end of the path, it has almost the same potential energy it had at the start. Except for the small loss due to friction, the energy of the system remains constant and is continually changing from one kind to the other. The total energy at any instant is, therefore, either the maximum value of the potential energy or the maximum value of the kinetic energy. If we deal with a case of true simple harmonic motion, the change from one form of energy to the other is similar to that just described. The total energy at any time is most easily expressed as the maximum value of the kinetic energy. The maximum velocity is $\frac{2\pi R}{T}$, or ωR, and $V^2 = \omega^2 R^2$.

$$\text{Kinetic energy} = \tfrac{1}{2} MV^2 = \tfrac{1}{2} M\omega^2 R^2.$$

51. Conservation of Energy.—In a system in which there are continual changes from potential to kinetic energy and *vice versa*, there is a slow diminution of mechanical energy and a development of heat through the operation of frictional forces. Heat is another form of energy, and later on we shall learn that heat may be measured in work units. There is every reason to believe that the loss of mechanical energy is exactly equal to the gain in heat when measured in energy units. Thus, the total energy suffers no loss due to the operation of the system. This idea, when expressed in general terms, is known as the Law of the Conservation of Energy. It may be stated as follows: In a system so constituted that it neither gives out energy to other systems nor receives energy from them, the total energy remains constant. This implies that energy can neither be created nor destroyed.

52 Power.—An agent, such as a steam engine, may do any amount of work if sufficient time is allowed for doing it, but its value as a machine will depend largely upon how quickly it can do a given amount of work. The rate at which an engine or a motor can do work is the most important item in its description.

The time rate of doing work is called power.

In the metric system the unit of power is the watt.

A watt is a time rate of doing work equal to one joule per second.

A kilowatt is 1,000 watts.

The English unit of power is the horse power. This unit was introduced by James Watt, and its name was supposed to give some idea of its magnitude.

The horse power is usually defined as a rate of doing work equal to 33,000 foot-pounds per minute, or 550 foot-pounds per second. This is subject to variations with latitude and with height above the sea level as other gravitational units are, so there is a tendency to state the definition in terms of the watt and thus relate it to the absolute system.

1 horse power (h.p.) = 746 watts.

Electrical power is usually measured in kilowatts and the abbreviation kw is used. Since the defining equation for power is $P = W/t$, we may clear the equation of fractions and express work as $W = Pt$. This means that work equals the power multiplied by the time during which the agent works, and a power unit multiplied by a time unit may be used as a work unit; thus the kilowatt-hour is the energy delivered in one hour by an agent working at the rate of one kilowatt. This is the energy unit in terms of which electric lighting bills are figured.

53. Problems.—Find the power of a steam pump that can pump 1,200,000 gallons of water per hour into a system of pipes in which the pressure is 40 pounds per square inch.

There are 231 cu. inches in a gallon; hence to pump a gallon of water into a pipe having a cross-section of one square inch would require a force of 40 pounds working through a distance of 231 inches, or 19.25 feet. The work done on a gallon would then be

40 × 19.25 = 770 foot-pounds.
1,200,000 gallons per hour = 20,000 gallons per minute.
Work per minute = 20,000 × 770 = 15,400,000 foot-pounds.
15,400,000 ÷ 33,000 = 466 horse power.

PROBLEMS

1. A tank is 24 ft. deep and has a capacity of 20,000 gallons. It stands with its bottom 50 ft. above the surface of the water in the well from which it is to be filled. If the water is pumped in through a pipe at the bottom of the tank, how much work is required to fill it? A gallon of water weighs 8.3 lbs.

2. In the preceding problem, if the water were pumped in at the top of the tank instead of at the bottom, how much work would be done in filling it?

3. Find the power transmitted by a belt moving with a velocity of 600 ft. per minute, passing around two pulleys. The difference in force in the upper and lower parts of the belt is 800 lbs.

4. A shell weighing 300 lbs. is fired from a gun weighing 50 tons with a velocity of 1,500 ft. per second. Find the velocity with which the gun begins its recoil. The mass of the powder is to be neglected. Will the gun or the shot be able to do more work before coming to rest; in what proportion?

5. Find the horse power of a pump that can pump half a million gallons of water an hour into a system of pipes in which the pressure is 40 lbs. per sq. inch. (There are 231 cu. in. in a gallon.)

6. If the water level at a certain dam is 8 meters above the turbines, and 50% of the energy of the water can be converted into electrical energy, how many grams of water per minute will supply the energy to operate a 40-watt tungsten lamp?

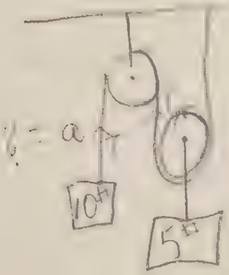
7. If a steam hammer weighing 800 lbs. were allowed to fall 2 ft. under the action of gravity, upon a piece of iron whose thickness is reduced by 1/8 of an inch as a result of the blow, what is the mean value of the force exerted upon the iron? (In this problem, we assume that the hammer suffers no deformation.)

8. A shell weighing 400 lbs. is fired from a gun 20 ft. long with a speed of 1,600 ft. per second. Find the average force produced by the explosion.

9. A bullet weighing 12 grams is fired horizontally into a block of wood whose mass is 5,000 grams. The block is suspended as a pendulum by long cords and the impact causes it to move with velocity of 70 cms per second. Find the speed of the bullet. Why cannot this problem be solved by equating the energy of the bullet to the energy of the block?

10. A locomotive rated at 1,600 h.p. should be able to exert what force while running 50 miles per hour?

$P = FV$



CHAPTER IV

MOMENTS OF FORCE AND CENTER OF MASS

54. Moment of a Force.—Consider a force applied to a rigid body as shown in Fig. 25, and suppose that the body is mounted upon a fixed axis through O perpendicular to the plane of the figure. The only motion that can then occur is a rotation about the axis. Let the vector AB represent the force acting upon the body. It is evident that if AB were to be moved so as to pass through O, it could produce no rotation of the body. Furthermore, the greater its distance from the axis O, the greater is its effect in producing an angular acceleration. Under the conditions assumed, the importance of the force in producing or changing the rotation of the body is evidently dependent upon two factors, the magnitude of the force and the perpendicular distance between the force and the point O. These factors are equally important, and we are led to define the moment of a force about a point as the product of the force and the perpendicular distance between the force and the point. A moment tending to produce a counter-clockwise rotation is considered as positive.

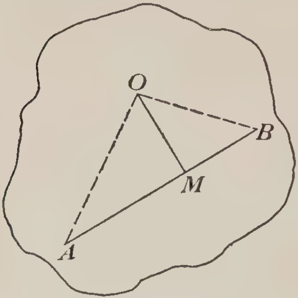


Fig. 25

If another force CD was applied to the body of Fig. 25, instead of AB, and if it was so placed that its moment about O was equal to that of AB, then the angular acceleration would be the same as that produced by AB. Two forces having equal and opposite moments about the same axis produce no effect upon the rotation about that axis and, in general, the angular acceleration is strictly proportional to the moment of the force producing it. These statements may be verified by experiment.

The moment of the force AB is $AB \times OM$, *i.e.*, twice the area of the triangle AOB. It is true, in general, that as a force may be represented by a straight line, so its moment about a point may be represented by twice the area of the triangle formed by joining the point to the extremities of the line representing the force. This relationship will be found to be useful in the next paragraph.

55. Moment of the Resultant of Several Forces.—In §20 it was pointed out that the process of finding the resultant of several forces is the same as the process of finding their vector sum.

The following important proposition is fundamental and should receive careful consideration. If several forces lie in the same plane and have a common point of application, the sum of their moments about any point in the plane is equal to the moment of the resultant about the same point. We shall first prove the proposition for the case of two forces. Let OA and OB, Fig. 26, be the two forces, and P the point about which the moments are taken. OC is the resultant.

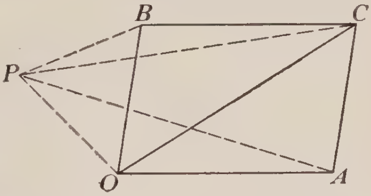


FIG. 26

$$\text{Moment of } OA = \text{twice area of triangle } POA.$$
$$\text{Moment of } OB = \text{twice area of triangle } POB.$$
$$\text{Moment of } OC = \text{twice area of triangle } POC.$$

All three of the triangles have the same base PO, and the altitude of POC equals the sum of the altitudes of the other two; hence the area of triangle POC equals the area of triangle POA plus the area of triangle POB. From this it follows that the moment of the resultant is equal to the sum of the moments of the component forces.

Another figure in which the three altitudes are shown may help to make evident the truth of the proposition. Draw Ak

parallel to PO, in Fig. 27. $Ck = Bx$ since these lines are homologous sides of equal triangles. $ky = Az$ since they are parallels intercepted between parallels. $Cy = Ck + ky$; hence by substitution, $Cy = Az + Bx$, which justifies the conclusion drawn above.

If the resultant of OA and OB is now considered in connection with another force, the proposition is established for these two forces and therefore for three forces, two of which are represented by the resultant OC. By continuing this course of reasoning, the proposition may be demonstrated for any number of forces.

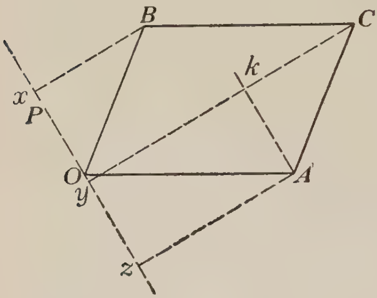


FIG. 27

In this demonstration there was no assumption made that would limit its application to any particular angle between the forces OA and OB; hence if the angle is made to approach zero, it will still be true that the moment of the resultant is equal to the sum of the moments of the individual forces. For this reason the proposition holds for the case of parallel forces.

56. Equilibrium Conditions.—If a body is to remain at rest when several forces are acting upon it, the first condition that must be satisfied is that the resultant of all the forces acting shall be zero. If this condition were not satisfied, there would be an acceleration of the center of mass of the body. While this is a necessary condition for equilibrium, it is not a sufficient condition. If, in addition, we impose a second condition that the sum of the moments of all the forces about any axis through the center of mass shall be zero, then not only the center of mass, but all other points of the body will remain at rest. Forces which satisfy the first condition may not satisfy the second. When both of these conditions are fulfilled, the sum of the moments of the forces about any axis will be zero.

57. Problem.—A crane has a boom 30 feet long and the ropes supporting the boom are connected to the mast at a point 40 feet above the hinged end of the boom. Consider the simple case in which the boom is horizontal. Find the force in the ropes supporting the boom when the load is 1,000 pounds and is suspended at a point 2 feet from the end of the boom.

We first find the perpendicular distance from the rope to the hinge on the boom, *i.e.*, the distance x in Fig. 28. Since the triangle AHB is a right triangle, the length of the rope is found to be 50 feet. From the similarity of the triangles

$$30 : 50 = x : 40,$$

from which $x = 24$ feet.

Since the boom is at rest we may apply the second condition of equilibrium and equate the positive moment acting upon it to the negative moment. By choosing the point H as the point around which the moments are computed, we simplify the problem, since the moment of the force applied to the boom at H becomes zero. Let F be the force in the rope; its moment about the hinge is Fx or 24 F pounds feet. This is a positive moment. The negative moment due to the load is 1,000 × 28 pounds feet. Equating the two moments,

$$24\,F = 28{,}000 \text{ and } F = 1{,}166 \text{ pounds.}$$

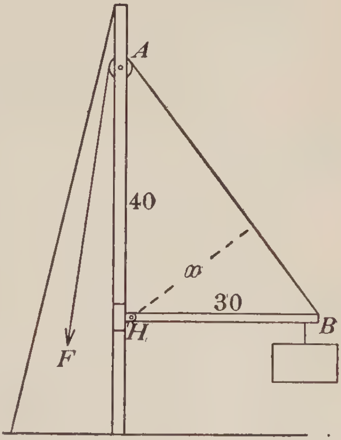


FIG. 28

The problem is solved in a similar manner if the boom is not horizontal.

58. Parallel Forces.—Two parallel forces may be considered as simply the limiting case of two concurrent forces when the angle between them approaches zero. Therefore the resultant is the sum of the two forces, and its moment about any point in the plane of the forces is equal to the sum of the moments of the components. Just as this proposition about moments was extended to include any number of concurrent forces, so it may be made to include any number of parallel forces.

The point about which the moments are taken may be any point in the plane; but the problem is simplified if, for that point, we take the point of application of the resultant. If this is done the moment of the resultant is zero, and hence the sum of the moments of the component forces is zero. Let F_1 and F_2 be two parallel forces and x_1 and x_2 their respective distances from the resultant, then

$$F_1x_1 + F_2x_2 = 0.$$

In order that this equation may be true, these two moments must be equal and opposite. Two cases should be considered; the forces may be in the same direction or in opposite directions.

(1) When the forces are in the same direction, as shown in Fig. 29, they can have opposite moments about a point only if the point lies between them; hence it appears that the resultant lies between the forces and its position is determined by the equation

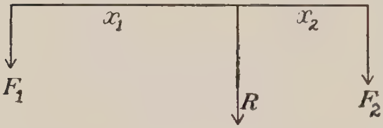


FIG. 29

$$F_1x_1 + F_2x_2 = 0.$$

Writing this equation in the form of a proportion, we have

$$F_1/F_2 = - x_2/x_1,$$

which states that the resultant divides the line between the two forces inversely as the magnitudes of the forces.

In an actual problem the distance AB between the forces must be given; hence we may substitute $AB - x_1$ for x_2, and there remains only one unknown quantity in the equation. Solving for x_1, we have the position of the resultant with reference to F_1 and F_2.

(2) In Fig. 30 the two forces are represented as oppositely directed. Keeping in mind the fact that the point of application of the resultant must be so located that the moments of F_1

and F_2 about it are equal and opposite, we see that this point cannot lie between the two forces, for the moments of F_1 and F_2 would then be directed in the same sense and could not cancel. However, there is a point, such as O, lying on the side of the greater force, around which the two moments are equal and opposite; and its position is given by the same equation as that used in the former case, *viz.*,

$$F_1x_1 + F_2x_2 = 0, \quad \text{or}$$
$$F_1x_1 + F_2(x_1 + AB) = 0,$$

in which there is only one unknown quantity. Solving for x_1, the position of R is found. In using the equation, the moments must be given their proper signs. In this particular case the moment of F_1 is positive and that of F_2 is negative.

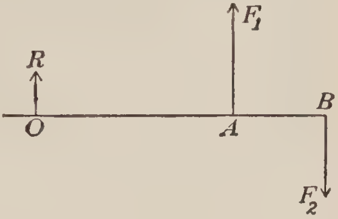


FIG. 30

59. Couples.—When the forces are parallel and oppositely directed, a special case of considerable interest arises from making the forces equal. Such a pair of forces is called a couple. Since the resultant of two oppositely directed forces is their difference, the resultant approaches zero as the forces approach equality. When the forces reach equality they still have a moment about any point in their plane, but they have no resultant. It thus appears that there is no single force which, acting alone, can produce the same result as a couple. The moment of the couple shown in Fig. 31 about the point O is

$$F_1 \times OA - F_2 \times OB.$$

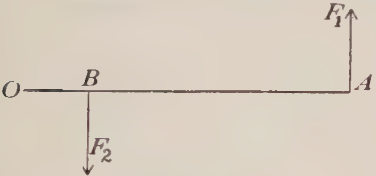


FIG. 31

Since $F_1 = F_2$, we may write F for either one and then factor the expression for the moment of the couple; thus

$$\text{the moment} = F(OA - OB) = F(BA).$$

Hence the moment of a couple is the product of one of the forces and the distance between the forces. It is of interest to note that the result is independent of the position of the point O, since neither OB nor OA appears in the final result. The conclusion to be drawn from this is that the moment of a couple is the same around all points in its plane. The idea that a couple is likely to produce rotation around a point halfway between the forces, rather than around some other point, is evidently not correct. The fact is that a body, initially at rest and free to move, would be set in rotation about an axis through its center of mass by any couple applied to it.

60. Problem.—An automobile weighing 3,000 pounds stands on a bridge 60 feet long. The center of mass of the automobile is 25 feet from one end of the bridge and 35 feet from the other end; what portion of its weight is supported by each pier of the bridge?

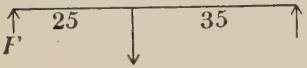


FIG. 32

The two forces supporting the automobile must have a resultant equal and opposite to its weight. The two forces are F and $3{,}000 - F$. The resultant divides the 60 feet into two parts inversely as the forces; hence $F : 3{,}000 - F = 35 : 25$, from which $F = 1{,}750$ pounds and $3{,}000 - F = 1{,}250$ pounds.

61. Resultant of Several Parallel Forces.—The proposition upon which the discussion of two parallel forces is based applies to any number of forces lying in the same plane; hence the location of the resultant of several parallel forces in one plane may be found from an equation similar to that of §58. We choose a point about which moments are to be taken wherever we like, and let x_1 be the distance from this point to the force F_1, x_2 the distance of the point from F_2, etc. Represent the distance from this point to the resultant by X; then

$$RX = F_1x_1 + F_2x_2 + F_3x_3 + \text{etc.}$$

Since $R = F_1 + F_2 + F_3 + \text{etc.}$,

$$X = \frac{F_1x_1 + F_2x_2 + F_3x_3 + \text{etc.}}{F_1 + F_2 + F_3 + \text{etc.}}$$

We use the Greek letter sigma to indicate the sum of a series of terms; thus we may write

$$X = \frac{\Sigma Fx}{\Sigma F}.$$

ΣF is read the sum of such terms as F. X is then the distance of the resultant of the several forces from the point about which the moments are taken.

If the forces are not all in one plane, the proposition may still be used, but with a slightly modified meaning of the letters. x_1, x_2, etc., now measure the distances of the forces from some plane of reference and X represents the distance of the resultant from this same plane.

62. Center of Gravity.—The center of gravity of a body is the point of application of the resultant of all the gravitational forces acting upon the body. Consider that the body is composed of mass particles of insignificant volume, and designate these masses as m_1, m_2, m_3, etc. The gravitational forces acting upon these particles are m_1g, m_2g, m_3g, etc. These gravitational forces are directed approximately toward the center of the earth and, if the body is not too large, they are so nearly parallel that the error is negligible if they are treated as parallel forces. To find the center of gravity, then, we use the equations of §61, *i.e.*,

$$X = \frac{m_1gx_1 + m_2gx_2 + m_3gx_3 + \text{etc.}}{m_1g + m_2g + m_3g + \text{etc.}}.$$

g is a factor common to all terms in both numerator and denominator and hence may be canceled. Indicating the summation by the symbol sigma, we have

$$X = \frac{\Sigma mx}{\Sigma m}.$$

X then indicates the distance of the center of gravity from a plane of reference. If three such planes of reference are used, we may find the distance of the center of gravity from each of the three planes, and hence determine its position in space.

In practical problems it frequently happens that only one plane of reference is necessary because the body is symmetrical about some axis, and it is known in advance that the center of gravity lies on this axis. A problem illustrating this appears in §66.

63. Center of Mass and Center of Inertia.—If the body represented in Fig. 33 is acted upon by a force F so placed that it produces translation only, every mass particle will move in the direction of F and all of them will have the same acceleration a. If we represent the masses of the particles by m_1, m_2, m_3, etc., the forces moving the particles will be m_1a, m_2a, m_3a, etc. Since these are merely the components of F, we may think of F as the resultant of all such forces as ma. The point of application of the resultant of all such forces is called the center of inertia, *i.e.*, the center of inertia is a point such that if a force is applied to the body at this point a pure translation without rotation will be the result. Representing the distance of the center of inertia from a plane of reference chosen parallel to the forces by X, we have, as in the preceding paragraph,

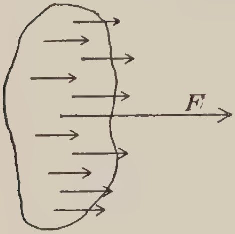


FIG. 33

$$X = \frac{\Sigma max}{\Sigma ma} = \frac{\Sigma mx}{\Sigma m}.$$

a, being a factor of every term of both numerator and denominator, will cancel. This is the same expression as that giving the distance of the center of gravity from a plane of reference; however, we remember that in deriving the expression for X in §62, we assumed forces to be parallel that meet near the center of the earth, while in the case now under discussion, the forces are accurately parallel. There is, therefore, a mathematical difference between the center of gravity and the center of inertia, although it is quite inappreciable for small bodies.

Since mass is the measure of inertia, the expression center of mass is merely another name for center of inertia. The name centroid is also frequently used to indicate this point.

64. Force Applied Not at the Centroid.—In Fig. 34 the force F is represented as applied to some point of the body other than its center of mass. In order to find out the effect of such a force, assume two opposite forces F' and F'', each equal to F, applied at the centroid in a direction parallel to the direction of F. The combined effect of these two forces is zero, so that the problem is in no way altered by their presence. F' taken alone would produce a translation in its own direction, while F and F'' constitute a couple which will produce rotation around the centroid. From this we conclude that a single force applied to a body so that its line of direction is not through the centroid will produce the same translation of the centroid as if it were applied at this point, but in addition it will produce a rotation around the centroid.

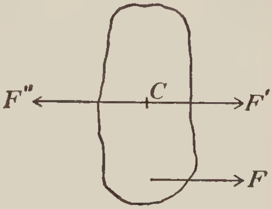


FIG. 34

65. Experimental Determination of Center of Gravity.—If a body is suspended on a string, it will adjust itself so that the center of gravity is in line with the string. This follows from the fact that if the resultant gravitational force did not coincide with the supporting force, there would be a moment tending to rotate the body and it would turn until the two forces did coincide. In this position, the center of gravity would be in the vertical line passing through the point of support. The body may now be turned and the supporting string fastened to some other point upon it. When it again comes to rest, the center of gravity will be in the vertical line through the point of support, and since it lies in both these lines, it lies at their intersection.

The same idea may be applied in a slightly different way. If a body is placed on a knife edge and its position adjusted until it balances, the center of gravity lies in a vertical line passing through the knife edge.

66. Problem.—A uniform shaft weighs 80 pounds and is 12 feet long. It has a pulley fastened at one end which weighs 40 pounds, and another pulley weighing 30 pounds is fixed 2 feet from the opposite end. Find the center of mass of the system.

Since the shaft is a cylinder and the pulleys are wheels symmetrical about their axes, the center of mass must lie on the axis of the shaft. Our problem then requires merely that we find the distance of the center of mass from either end of the shaft. The plane of reference is taken at right angles to the axis of the shaft. Suppose we choose it three feet from the left end of the shaft, as illustrated in Fig. 35. The shaft may be treated as if its mass were concentrated at its middle point, *i.e.*, 9 feet from the plane of reference.

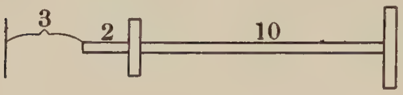


FIG. 35

$$X = \frac{80 \times 9 + 30 \times 5 + 40 \times 15}{80 + 30 + 40}$$
$$= 9.8 \text{ feet from the plane of reference}$$
$$= 6.8 \text{ feet from the left end of the shaft.}$$

The problem is somewhat simplified if the plane of reference is chosen through one end of the shaft. It is suggested that the student work the problem with the plane of reference first through one end of the shaft and then through the other end. In both cases the result should agree with that just obtained.

PROBLEMS

1. A long steel bar weighing 100 lbs. is suspended on a hinge at its upper end so that it may swing as a pendulum. A force is applied at its lower end perpendicular to the length of the bar. The bar is thus drawn aside through an angle of 30°. Find the magnitude of the force to hold the bar in this position. Solve by the method of moments.

2. In the preceding problem, if the force were horizontal, find its magnitude.

3. A trailer having two wheels has its center of gravity at a height h from the ground. The distance between its wheels is a. It runs around a curve on a horizontal pavement. Prove that it will overturn if its velocity V is greater than $\sqrt{\frac{gaR}{2h}}$, R being the radius of the curve.

4. The boom of a crane is 18 ft. long, and the tackle is fastened to the vertical pole 24 ft. above the boom. A weight of 1 ton is to be lifted. Find the force in the ropes when the boom is horizontal. Solve by the method of moments.

5. Solve the preceding problem by a graphical method, and find also the force transmitted by the boom. (Make the drawing to scale and form a parallelogram of forces.)

6. A force of 15 lbs. acts vertically upward, and another of 10 lbs. acts vertically downward and is 20 inches from the first force. What is their resultant and where is it applied?

7. A bridge weighing 14,000 lbs. is 30 ft. long. An automobile weighing 4,000 lbs. stands with its center of gravity 12 ft. from one end. Find the force on each pier.

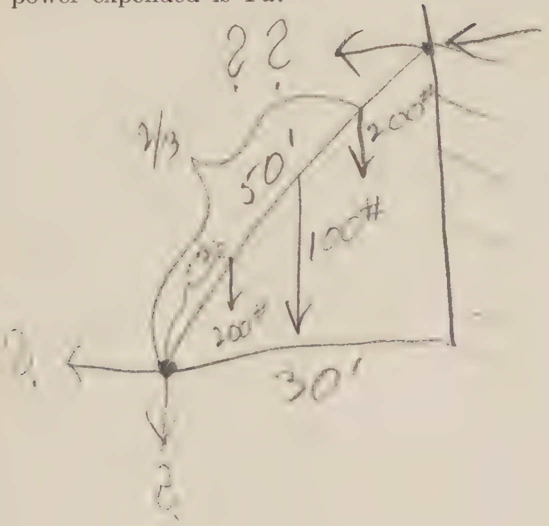
8. A steel shaft 6 ft. long weighs 40 lbs. It has a pulley at one end weighing 15 lbs. and another 18 inches from the other end weighing 20 lbs. Find the center of mass of the system.

9. A steel shaft 6 ft. long is 3 inches in diameter; it is put into a lathe and turned down at one end to a diameter of 2 inches throughout 2 ft. of its length. Find its center of mass.

10. A thin sheet of metal in the form of a square 1 ft. on a side is divided into four squares by drawing two lines upon it. One of the four squares is cut away; find the center of mass of the remaining part.

11. One corner of a square sheet of paper is folded down so as to coincide with the center of the square. Taking the side of the square as 1 ft. long, find the distance of the center of mass of the paper from the center of the square.

12. If a wheel is turned by a torque T at an angular velocity ω, prove that the power expended is $T\omega$.



CHAPTER V

MACHINES

67. Machines.—A machine is a device for doing work. It is used because it offers a certain advantage in the application of force. It may be that a greater force can be exerted by means of the machine than could be exerted without it. Most of the simple machines offer an advantage of this kind. Occasionally a considerable motion is desired, and it is advantageous to exert a large force through a short distance in order to obtain a small force operating through a long distance. The mechanism of the forearm is an illustration of this: the hand moves through a distance that is much greater than the amount of the contraction of the biceps. There might be a considerable advantage in merely changing the direction of a force employed in doing a certain piece of work. The advantage in raising a flag by pulling downward on the rope instead of upward is obvious. However, because the advantage in using the machine is so often merely due to the fact that a greater force is possible through its use, it has become customary to define the mechanical advantage of a machine as the ratio of the force exerted by the machine to the force applied to the machine. The load is the force against which the machine works and is equal and opposite to the force exerted by the machine. Call the load L and the force that operates the machine F; then the mechanical advantage of the machine is L/F.

Newton named six devices which he regarded as fundamental in the construction of machinery. Most complicated machinery, if analyzed, would be found to be merely a combination of some of these devices. They are the lever, the pulley, the wheel and axle, the inclined plane, the wedge, and the screw. These six have come to be known as the six simple machines. It is assumed that the student is somewhat familiar with these and it will therefore not be necessary to discuss each in detail.

The problem of determining the actual mechanical advantage of a given machine is a laboratory problem requiring an experimental method. Usually it consists in measuring L and F under working conditions and taking their ratio.

An ideal machine is one in which there is no friction. Any of the simple machines may be treated as if it were ideal, in which case it is possible to express the mechanical advantage in terms of parts of the machine. For purposes of machine design, it is customary to calculate the mechanical advantage of the ideal machine and then to make allowance for the effect of friction, the allowance being determined by previous experience.

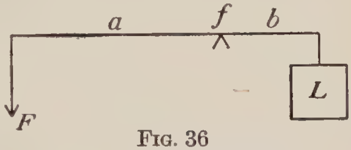


FIG. 36

68. Mechanical Advantage of the Ideal Machine.—Several methods of approaching this problem are possible. Some are only applicable in a few cases, but one in particular can be used in dealing with any machine.

The principle of moments may be applied satisfactorily in finding the mechanical advantage of a lever. In Fig. 36 there is represented what is usually called a lever of the first class. a and b are the lever arms and f the fulcrum. For equilibrium,

$$Fa = Lb \text{ and } \frac{L}{F} = \frac{a}{b}.$$

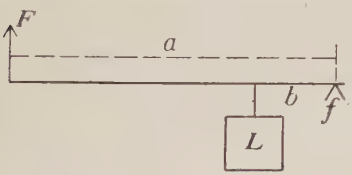


FIG. 37

The same demonstration applies to levers of the second and third classes shown in Figs. 37 and 38. The result is stated as follows: The mechanical advantage of a lever is the inverse ratio of its arms.

The student can easily apply the principle of moments to the wheel and axle, as shown in Fig. 39. The wheel and the axle are rigidly fastened together and turn about an axis through the center.

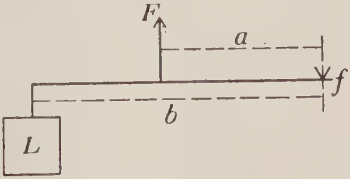


FIG. 38

A second method of finding the mechanical advantage of a machine is by direct comparison of the forces. This method may be applied with advantage to the inclined plane and to systems of pulleys. In Fig. 40 we have represented an inclined plane with the force applied parallel to the plane. By resolving L into two components, one parallel to the plane and the other perpendicular to the plane, we see at once that F must be equal and opposite to the component parallel to the plane. No work is done against the other component since it is perpendicular to the plane. Triangles *abc* and *def* are similar and hence

$$L/F = ac/bc,$$

i.e., the mechanical advantage of an inclined plane, when the applied force is parallel to the plane, is equal to the ratio of the length of the plane to its height.

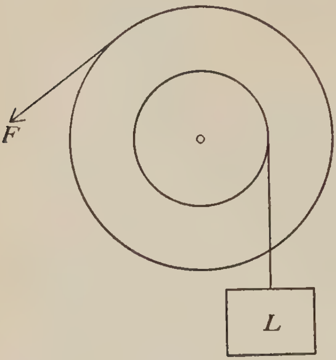


Fig. 39

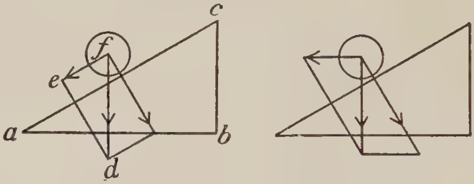


Fig. 40

If the applied force is horizontal the diagram is that of Fig. 40 (right), and it is easily seen that the mechanical advantage is equal to the ratio of the base of the plane to its height.

69. Pulleys.—Systems of pulleys having a single rope may be arranged, as in Fig. 41, with the end of the rope fastened to the stationary block or, as in Fig. 42, with the end of the rope fastened to the movable block. Since we assume that there is no friction in the system, the force in the rope is uniform through-

F Work S S T

out its length, and the force supporting the load is the force in the rope multiplied by the number of parts of the rope supporting the load. In Fig. 41 the number of parts of the rope supporting the load is four, and in Fig. 42 it is five. Then the load is four times the applied force in the one case, and five times the applied force in the other. In general, the mechanical advantage of a system of pulleys employing one rope is equal to the number of parts of the rope supporting the load. In this discussion it is assumed that the parts of the rope are parallel and that the weight of the lower pulley block may be neglected. In practice the blocks are usually made so that all the pulleys are on the same axis. They are shown in the figures on separate axes and of different sizes for the sake of clearness.

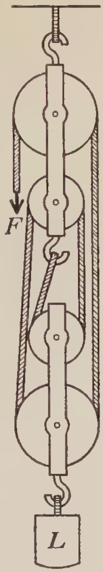


FIG. 41

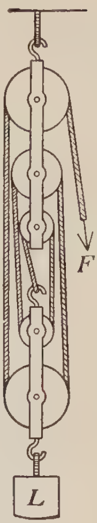


FIG. 42

When a system is constructed with more than one rope, as in Fig. 43, the force in one rope is in general different from the force in the other so that the mechanical advantage cannot be determined by merely counting the parts of the rope that support the load. It is suggested that the student study Fig. 43 and determine the mechanical advantage of the system.

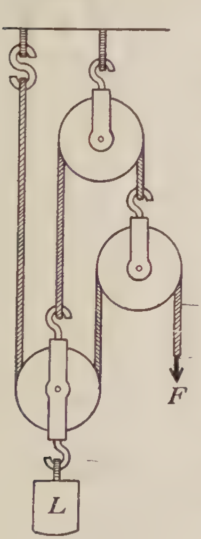


FIG. 43

We shall next discuss and illustrate a third method of determining the mechanical advantage of a machine. This method is more useful than either of the other two because it may be applied to any machine. We are still dealing with the ideal machine. The assumption that there is no friction is also an assumption that there is no energy lost in heat, and consequently the work done by the machine equals the work done on the machine. If the applied force works through a distance S and the load is at the same time moved through a distance s, the work done on the machine equals FS and the work done by the machine equals Ls; putting these equal, $FS = Ls$ and the

$$\text{Mechanical advantage} = \frac{L}{F} = \frac{S}{s}.$$

In most cases it is possible to determine from the construction of the machine this ratio S/s and hence determine the mechanical advantage. We shall apply the method to several machines by way of illustration, but it should be understood that the student ought to be able to apply it to almost any machine whose mechanism he understands.

In the system shown in Fig. 41, if the force F moves downward through a distance S, each part of the rope supporting the load will be shortened by an amount equal to 1/4 of S and the load will be lifted through a distance $s = S/4$. The mechanical advantage is then $S/s = 4$. In Fig. 42, when F moves a distance S, the load is lifted 1/5 of S and the mechanical advantage is 5. We thus arrive at the conclusion that was reached earlier by another method, *viz.*, that the mechanical advantage of a system of pulleys employing one rope is equal to the number of parts of the rope that support the load.

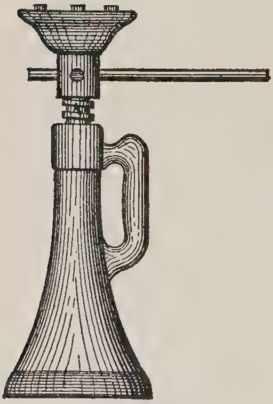


FIG. 44

70. The Screw.—The screw as a machine is a device for lifting very heavy weights. When a house is to be lifted from its foundation, a number of jackscrews constitute the only machines necessary for doing the work. If the screw is constructed as shown in Fig. 44, the load rests upon a piece of iron which bears upon the top of the screw, and the screw turns in a long nut having a broad base and resting upon some solid foundation. As the screw is turned through one revolution the load is lifted through a distance equal to that between two corresponding parts of adjacent portions of the thread. This distance is called the pitch of the screw and may be represented by d. The force is applied in a horizontal direction to a long bar whose length we designate as R. The work done by F in one revolution is $2\pi RF$ and the work done on the load is Ld. Treating the machine as if it were ideal,

$$2\pi RF = Ld,$$

from which we determine the mechanical advantage to be

$$\frac{L}{F} = \frac{2\pi R}{d},$$

and we have the mechanical advantage in terms of the length of the bar and the pitch of the screw. Since $2\pi R$ is very large as compared with d, the mechanical advantage is very large.

71. The Differential Pulley.—This mechanism is frequently called the chain hoist. It is used in machine shops for raising heavy weights and is often found in boat houses for lifting boats out of the water. If the apparatus is not familiar to the reader, it may be easily understood from the diagram in Fig. 45. What is pictured here as a rope is in reality an endless chain whose links fit into projections on the pulleys so that slipping is impossible. At the top are two pulleys rigidly fastened together and having unequal radii. Call the larger radius R and the smaller one r. The load hangs upon the axis of the lower pulley and the force F is applied to the chain as shown in the figure. When the force F causes the upper pulleys to make one revolution, a length of chain equal to $2\pi R$ is wound on to the larger pulley, and at the same time a length equal to $2\pi r$ is reeled off from the smaller one. The part of the chain supporting the lower pulley is thus

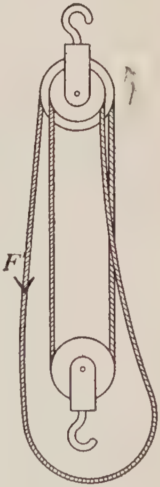


Fig. 45

shortened by an amount equal to the difference between these two lengths, or $2\pi R - 2\pi r = 2\pi (R - r)$. The load is lifted through half this distance, or $\pi (R - r)$. The force F moves through a distance equal to $2\pi R$; hence the mechanical advantage is

$$\frac{S}{s} = \frac{2\pi R}{\pi(R - r)} = \frac{2R}{R - r}.$$

It is thus apparent that by making the two upper pulleys nearly the same size, the mechanical advantage may be made very great.

72. Efficiency.—The efficiency of a machine is the ratio of the useful work performed to the total work done on the machine. This amounts to the same thing as defining the efficiency as the ratio of the useful power developed to the power used in operating the machine. We have been discussing the ideal machine in which the useful work was considered as equal to the work done on the machine; thus the efficiency of the ideal machine is 100%. The actual machine, in which energy is converted into heat because of the operation of frictional forces, must necessarily have an efficiency less than 100%. For this reason also, the mechanical advantage of the actual machine is less than that of the ideal machine. Let us now use L and F to represent respectively the load and the applied force in an actual machine. As before, s and S represent the respective distances through which these forces move. Let E equal the efficiency; then by the definition of efficiency, $E = \frac{Ls}{FS}$. This may be written

$$E = \frac{L/F}{S/s}.$$

But L/F is the actual mechanical advantage of the machine and S/s has been shown to equal the mechanical advantage of the ideal machine, from which it follows that the efficiency of a machine is equal to the ratio of its mechanical advantage to the mechanical advantage of the same machine when considered as an ideal machine. This statement is true for all the simple pieces of mechanism that have been under consideration in this chapter, but it ought not to be considered as a definition of efficiency since this term is a broad one and is used in connection with many devices and processes not involving mechanical advantage at all. For example, it is perfectly proper to speak of the efficiency of an electric lamp, which is not a machine and has no mechanical advantage. The ratio of the useful energy to the total energy supplied in a given time is a definition of efficiency that applies to all cases where the word is used in a scientific sense. It applies equally well to the electric lamp or to a lever or a system of pulleys.

73. Machines in Series.—Two or more machines may be used in such a way that the load of one becomes the operating force of another. The rope leading upward from the drum of the winch shown in Fig. 46 may be the rope that operates a system of pulleys. If F is the force applied to the crank and L is the force transmitted by the rope, the mechanical advantage of the winch is, by definition, $M = L/F$. Let M' be the mechanical advantage of the pulley system alone and L' the load on the pulleys, then $M' = L'/L$ since L is the force applied to the pulley system. Multiplying the equations member by member,

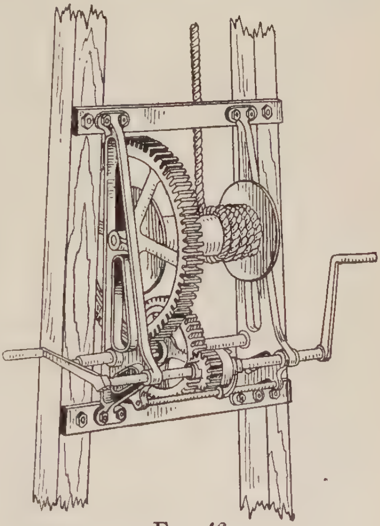


FIG. 46

$$MM' = \frac{L}{F} \cdot \frac{L'}{L} = \frac{L'}{F}.$$

But L'/F is, by definition, the mechanical advantage of the entire system, *i.e.*, of the two machines in series.

Suppose the efficiency of the winch to be E and that of the pulley system to be E'. If an amount of work W is done on the crank, only the fraction E of this work is done on the rope leading to the pulley system. Of this amount of work EW done on the pulley system, only the fraction E' is done on the load, hence $E'EW$ equals the useful work and W is the total work done on the machine. The ratio of these two quantities is, by definition, the efficiency of the machine. $E'E$ then represents the efficiency of the machines in series.

In general, it is true that the mechanical advantage of several machines operating together in series is the product of the individual mechanical advantages and the efficiency of the entire system is the product of the several efficiencies.

As an application of some of the foregoing principles, the student may now compute the mechanical advantage of the winch shown in Fig. 46. Make an estimate of the number of teeth on each gear and assume values of the length of the crank and the radius of the drum. The mechanical advantage is expressed in terms of these six quantities.

74. Self-Locking Machines.—A self-locking machine is one which, on account of friction, does not run backward while supporting the load if the applied force F is reduced to zero.

Let F be the force to balance a load L in the case of the ideal machine. Now if the machine does not run backward when the applied force is reduced to zero, the effect of friction in the machine is at least as great as the effect of the force F on the ideal machine. Suppose further that the force of friction is only just sufficient to prevent the machine from running backward; assume also that the frictional resistance to the motion of the machine is the same whether it is running forward or backward. On the basis of these two assumptions, when a load L is moved, a force $2F$ will be needed, half of which is used in overcoming frictional resistance and half of which balances the load. It thus appears that just twice as much work is done on the machine as is done by the machine, and hence its efficiency is 50%. When the effect of friction is greater than just enough to make the machine self-locking, the efficiency is less than 50%. The screw and the differential pulley, as usually built, are examples of self-locking machines.

PROBLEMS

1. A body when weighed in the right-hand pan of a beam balance, apparently weighed 20.4 grams, and when weighed in the left pan, it had an apparent weight of 20.8 grams. Find the true weight of the body and the ratio of the arms of the balance.

2. Make drawings of systems of pulleys whose ideal mechanical advantages are respectively 4, 5, 6, and 7, a single cord being used in each case.

3. Make two drawings of pulley systems in which two or more cords are used, and determine the mechanical advantage in each case.

4. A certain system of pulleys, when considered as an ideal machine, has a mechanical advantage of 5. Make a sketch of the system. If, because of friction, the force in the rope is diminished by 10% at each pulley, find the efficiency of the system. What does the mechanical advantage become when the frictional resistance is considered?

5. A paved road makes an angle of 18° with the horizontal. On another paved road the force required to propel a vehicle is only one-third as great as on the first road. What is the angle the second road makes with the horizontal?

6. A pulley system having a mechanical advantage of 2 has a small platform weighing 20 lbs. suspended from the lower pulley block. A man sits on the platform and pulls himself up at a uniform speed by pulling downward on the rope. The man weighs 180 lbs. and the machine is considered as an ideal machine. Illustrate the problem by a sketch and find the force the man exerts upon the rope.

7. A single fixed pulley offers a frictional resistance to motion such that the force in the rope on one side of the pulley is only 90% of the force on the other side. What is the efficiency? The same pulley is now used as a movable pulley; what is the efficiency?

8. The pitch of a jack screw is 1/2 inch; the screw is turned by a lever 30 inches long. Treating this machine as if it is just barely self-locking, find its mechanical advantage.

9. A hoisting apparatus consists of a drum 8 inches in diameter upon which the rope is wound and a gear having 96 teeth keyed to the shaft carrying the drum. This gear meshes with another having 12 teeth which is turned by a crank 16 inches long. Sketch the machine and determine its mechanical advantage, treating it as an ideal machine. Would the result be more or less if the diameter of the rope were taken into consideration?

10. A differential pulley is built with the two wheels in the upper block having 24 and 25 cogs respectively. Find the ideal mechanical advantage. Find the maximum mechanical advantage it could have if it were self-locking. Under the latter assumption find the weight that could be lifted by applying 100 lbs. to the chain.

CHAPTER VI

ROTATION OF A RIGID BODY

75. Definition.—A rigid body is one whose shape is not altered by any forces that are applied to it.

Such a body is only an ideal conception, for rigid bodies do not exist. However, for immediate practical purposes, we may treat almost any solid body as if it were rigid without appreciable error. Such bodies as flexible springs or pieces of rubber ought not to be regarded as rigid under any conditions, and of course fluids have no rigidity.

76. Moment of Inertia.—When a rigid body is rotating about a fixed axis, the mass particle that is farthest from this axis has the greatest velocity, and the velocities of particles nearer the axis are less. The momentum of a mass particle is the product of its mass and its velocity. The momentum of a particle in the rotating body depends upon its distance from the axis of rotation, being equal to

$$mv = m\omega r.$$

The product of a force and its distance to a point we have learned to call the moment of the force about the point. So the product of the momentum of a mass particle and its distance to the axis around which it revolves might be called its moment of momentum. However, the term angular momentum, rather than moment of momentum, is the one usually applied to this important quantity. Moreover, just as the moment of a force is the measure of its importance in producing or stopping rotation, so the moment of momentum, or angular momentum, of the particle is the

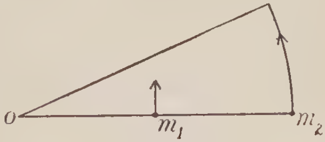


FIG. 47

measure of its importance in maintaining rotation against a resisting force moment.

$$\text{Angular momentum} = mvr = m\omega r^2.$$

This equation applies to the mass particle; if we wish to find the angular momentum of the body of which it is a part, the angular momenta of all the individual particles must be added. This addition may be indicated by using the summation sign, sigma:

$$\Sigma mvr = \Sigma m\omega r^2.$$

ω is the angular velocity of any line of the body perpendicular to the axis, and is a factor common to every term of the series. We may factor the terms and place the ω outside the summation sign, thus:

$$\Sigma mvr = \omega\Sigma mr^2.$$

The sum of such terms as mr^2 is evidently independent of the motion of the body, and is a property of the body which is determined by its mass and the distribution of the mass around the axis. It is called the *moment of inertia* of the body. There is no advantage in defining the moment of inertia in words which avoid the use of the formula; it is sufficient to say that the moment of inertia of a body about a given axis is the sum of such terms as mr^2, m being the mass of an individual particle and r its distance from the axis. This quantity, known as a moment of inertia and designated by the letter I, appears in every proposition and in every problem related to the rotation of bodies, as well as in some problems not concerned with rotation.

The student can find in his own experience some illustrations of the ideas we are now considering. Imagine two flywheels to be used on engines. Suppose that they are both made with the same amount of iron, but one has twice the diameter of the other. Most of the iron we assume is in the rim, and that which is not we may neglect. If both wheels make the same number of revolutions per minute, ω is the same in the two cases.

The linear velocity of the iron in the large wheel is double that of the iron in the small wheel. The momentum of each particle of the large wheel is twice as great as the momentum of a particle of equal mass in the small wheel. The angular momentum is four times as great in the first case as in the second. The two wheels are evidently very different in their mechanical behavior, not because their masses are different but because the distribution of the mass in the one case is different from that in the other. The larger wheel, having the larger moment of inertia, is more effective in resisting fluctuation in the driving moment, and hence more effective in maintaining a nearly constant speed.

The determination of the moment of inertia of a body having an irregular shape is a laboratory problem and need not be discussed here.

For bodies having regular geometric forms, the moments of inertia can be found by mathematical processes. These problems receive considerable attention in the courses in mathematics and consequently should be omitted from this course.

There is one body whose form is such that its moment of inertia can be immediately expressed without the necessity of a mathematical treatment. It is a thin ring or a thin hollow cylinder. A circular hoop made of wire or a length of a stovepipe will illustrate what is meant. All the mass particles are at practically the same distance from an axis through the center of the hoop and at right angles to its plane. The same is true of the hollow circular cylinder, if the axis is the axis of the cylinder. For this case, then, in the expression Σmr^2, all the radii are equal and r^2 is a common factor appearing in every term of the series. Removing this common factor,

$$I = r^2\,\Sigma m;$$

but $\Sigma m = M$, the total mass of the body; r becomes R, the radius of the cylinder, hence

$$I = MR^2.$$

It must be remembered that r represents the distance from the axis of rotation to the mass particle and does not in general represent the radius of a body. When the distances to the particles are different, it becomes necessary to resort to integration.

77. Moments of Inertia for Some Bodies.—For those bodies whose motions we shall study in this course, the moments of inertia are given in the following list. The list is here for reference and need not be memorized.

Uniform thin rod, axis through the middle and at right angles to the length of the rod, L = length of the rod:

$$I = \frac{ML^2}{12}.$$

Thin rectangular plate, axis through the center of mass at right angles to the surface of the plate, a and b are the two dimensions:

$$I = \frac{M(a^2 + b^2)}{12}.$$

Thin ring or circular cylindrical shell, axis being the axis of the cylinder, radius = R:

$$I = MR^2.$$

Solid circular cylinder, axis being the axis of the cylinder, radius = R:

$$I = \frac{MR^2}{2}.$$

Thick hollow cylinder, axis through the center of mass parallel to elements of cylindrical surface; larger radius = R, smaller radius = r:

$$I = \frac{M(R^2 + r^2)}{2}.$$

Sphere, axis a diameter, radius = R:

$$I = \frac{2MR^2}{5}.$$

78. Moments of Inertia About Parallel Axes.—We shall now demonstrate an important proposition that makes it possible to determine the moment of inertia for any axis, provided we know the moment of inertia for a parallel axis through the center of mass. In Fig. 48, O represents an axis perpendicular to the plane of the diagram and through the center of mass of a body. C is another axis parallel to the first and at a distance d from it. Let I_o be the moment of inertia about the axis through the center of mass, and I the moment of inertia about the parallel axis. I_o and d are known quantities, and we are to find an expression for I in terms of these. Let m be any mass particle in the body, and s its distance from the axis C, and r its distance from the axis O. By a familiar proposition in trigonometry,

$$s^2 = r^2 + d^2 + 2dr \cos \theta.$$

If we multiply each term of the equation by m, we have

$$ms^2 = mr^2 + md^2 + 2mdr \cos \theta.$$

For every mass particle in the body we might write an equation similar to this, and the figure to which it would refer would lie in a plane parallel to the plane of Fig. 48. Let us assume that we add all these equations member by member and indicate the result in the usual way:

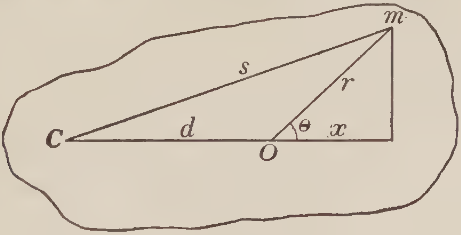


FIG. 48

$$\Sigma ms^2 = \Sigma mr^2 + \Sigma md^2 + \Sigma 2mdr \cos \theta.$$

Σms^2 is the moment of inertia which we are to determine, *i.e.*, I. Σmr^2 is the moment of inertia about the axis through the center of mass, *i.e.*, I_o. The expression Σmd^2 represents a series, each term of which contains the factor d^2. Taking out the common factor, the series becomes $d^2\Sigma m$, and the sum of such terms as m is M, the total mass of the body. From this it is apparent that $\Sigma md^2 = Md^2$. The last term of the equation needs rather careful consideration. The 2 and the d are common factors in the series. Imagine a plane passed through O perpendicular to d; then $r \cos \Theta$ equals x, the distance of the mass particle from this plane. This will be true whether we choose a particle lying to the right of the plane or to the left of it in the figure. Substituting x for $r \cos \Theta$ in the last term, $\Sigma 2mdr \cos \Theta = 2d\Sigma mx$. In §63 we found that $\frac{\Sigma mx}{\Sigma m} = X$ = the distance of the center of mass from the plane of reference from which the distances x are measured. Then $\Sigma mx = X\Sigma m = MX$. In this discussion the plane from which X is measured passes through the center of mass and therefore $X = 0$; then $MX = 0$, and the last term in the equation is zero. The equation thus reduces to

$$I = I_o + Md^2.$$

To illustrate the use of this relationship, suppose we wish to know the moment of inertia of a solid cylinder about an axis which lies in the surface of the cylinder. Consulting the list in §77, we find that I_o for the solid cylinder equals $\frac{MR^2}{2}$. The distance d between the two axes is the radius R, hence

$$I = \frac{MR^2}{2} + MR^2 = \frac{3MR^2}{2}.$$

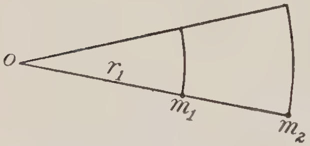


FIG. 49

79. Kinetic Energy of Rotation.—In Fig. 49, o represents an axis perpendicular to the plane of the figure about which a body rotates. m_1 and m_2 are two of the mass particles of which the body is composed. r_1 and r_2 represent the respective distances of the particles from the axis. If the body rotates with angular velocity ω, the linear velocity of m_1 is $v_1 = \omega r_1$, and the linear velocity of m_2 is $v_2 = \omega r_2$. K. E. may be used to designate kinetic energy.

$$\text{K.E. of } m_1 = \frac{m_1 v^2{}_1}{2} = \frac{m_1 \omega^2 r^2{}_1}{2}.$$

$$\text{K.E. of } m_2 = \frac{m_2 v^2{}_2}{2} = \frac{m_2 \omega^2 r^2{}_2}{2}.$$

Let us imagine that equations like these are written for all the mass particles of the body, and that these equations are added. The sum of the first members is simply the total kinetic energy of the body, so that the resulting equation is

$$\text{Total K.E.} = \tfrac{1}{2}(m_1\omega^2 r^2{}_1 + m_2\omega^2 r^2{}_2 + m_3\omega^2 r^2{}_3 + \text{etc.})$$

Taking out the common factor ω^2, the equation becomes

$$\text{Total K.E.} = \tfrac{1}{2}(m_1 r^2{}_1 + m_2 r^2{}_2 + m_3 r^2{}_3 + \text{etc.})\omega^2.$$

The quantity within the parentheses is the sum of such terms as mr^2 and is the moment of inertia of the body; therefore

$$\text{K.E.} = \frac{I\omega^2}{2}.$$

This is measured in absolute units, since $\frac{mv^2}{2}$, the expression from which it was derived, measures the energy in absolute units.

80. Problem.—A grindstone weighing 200 lbs. is 3 ft. in diameter. Find its kinetic energy when it is rotating at the rate of 100 revolutions per minute.

The stone is a solid cylinder and the axis is through the center of mass; therefore its moment of inertia is:

$$\frac{MR^2}{2} = \frac{1}{2} \times 200 \times \frac{9}{4} = 225 \text{ pounds ft.}^2$$

$$\omega = \frac{2\pi 100}{60} \text{ radians per second and } \omega^2 = 109.66.$$

$$I\omega^2/2 = 12{,}336 \text{ foot-poundals or } 385.5 \text{ foot pounds.}$$

81. Energy of Rolling Bodies.—In §7 mention was made of the fact that, when the axis of rotation is continually changing, as in the case of a rolling body, there are two ways of handling a given problem. The determination of the kinetic energy of a rolling cylinder furnishes an illustration of these two ways of treating problems of this kind. A solid cylinder of mass M and radius R rolls along a horizontal floor, the center having a velocity V. We shall find the kinetic energy first by considering the motion as rotation about that element of the cylindrical surface that is in contact with the floor. The moment of inertia of the cylinder for this axis is $I_o + Md^2$ or $MR^2 + \frac{MR^2}{2} = \frac{3MR^2}{2}$. The angular velocity ω is V/R, since the center of the cylinder is at a distance R from the axis.

$$\text{K.E.} = \frac{1}{2}I\omega^2 = \frac{1}{2} \cdot \frac{3}{2}MR^2 \cdot \frac{V^2}{R^2} = \frac{3}{4}MV^2.$$

We shall next solve the problem by resolving the motion into a translation of the centroid and a rotation about an axis through the centroid. The angular velocity about this axis is $\omega = V/R$, and the energy of this rotation is

$$\frac{1}{2}I\omega^2 = \frac{1}{2} \cdot \frac{1}{2}MR^2 \cdot \frac{V^2}{R^2} = \frac{1}{4}MV^2.$$

To this we must add the kinetic energy of translation, $MV^2/2$, and we have

$$\text{total K.E.} = \frac{1}{4}MV^2 + \frac{1}{2}MV^2 = \frac{3}{4}MV^2,$$

which is the same result as that obtained by the other method.

82. Angular Acceleration.—A point moving on a circular arc always has an acceleration toward the center of the arc, as we found in §32; but it may at the same time have an acceleration along the arc. The two accelerations are at right angles and may be treated as separate quantities. In the following discussion, let a be only the acceleration along the arc. If the acceleration is constant and the velocity increases from V_o to V in t seconds, then $a = \frac{V - V_o}{t}$. Since $V = \omega r$,

$$a = \frac{V - V_o}{t} = \frac{\omega r - \omega_o r}{t} = \frac{(\omega - \omega_o)r}{t}.$$

$\frac{\omega - \omega_o}{t}$ is the time rate of change of angular velocity and is accordingly called angular acceleration. We designate it by the letter alpha (α). Thus we show that $a = \alpha r$. This is an important relationship. The student should not regard it as a mere algebraic equation, but should try to appreciate its physical significance.

83. Torque and Angular Acceleration.—The term *torque* signifies the product of a force into the perpendicular distance between the force and a point, *i e.*, it has precisely the same meaning as the moment of a force. Just as a force when acting upon a body gives it a linear acceleration, so a torque, when acting upon a body that is free to turn about a fixed axis, gives it an angular acceleration. In the case of the linear motion, the relation between the force and the acceleration is given by the equation $F = ma$. Our problem now is to find the relation between the applied torque and the resulting angular acceleration.

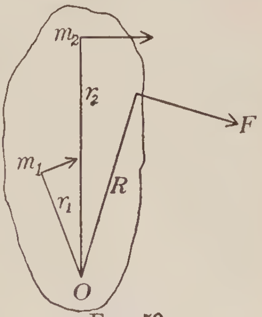


FIG. 50

Fig. 50 represents a body free to turn about an axis O perpendicular to the plane of the figure. A force F is shown having a lever arm of length R; hence the torque due to this force is FR. Assume that this torque produces an angular acceleration α. Any mass particle as m_1 at a distance r_1 from the axis will have a linear acceleration a equal to αr_1, and the force required to

moment of a couple
when 2 = + || and opposite forces act
on an object a couple is formed, Torque

give it this acceleration is m_1a_1 equal to $m_1\alpha r_1$. The moment of force required to produce the motion of this mass particle is

$$m_1\ \alpha r^2{}_1.$$

The torque FR furnishes this moment and also the moment required to move all the other mass particles, hence

$$FR = \Sigma m\alpha r^2.$$

The same α occurs in every term of the series, so we may write

$$FR = \alpha\Sigma mr^2.$$

We recognize Σmr^2 as the moment of inertia of the body about an axis through O and hence may substitute I. Then

$$FR = I\alpha.$$

It will be observed that, in deriving this formula, we substituted ma for force, which signifies that the force is expressed in absolute units; hence in using this relationship we must remember to express the force in dynes or poundals. The preceding topics have been studied principally in order to lead up to this formula. It is the foundation of the dynamics of rotation, and furnishes the solution of a whole series of problems related to the action of forces when rotation is produced.

This is an appropriate place to call the student's attention to §§ 152 and 153, which are related to the whole subject of the Mechanics of Solids and are at the end of this part of the subject.

84. Problem.—Assume the grindstone of § 80 to turn without friction. A cord is wound around it and a force of 10 lbs. is applied to the cord. The torque thus applied causes the stone to rotate and the cord to unwind with an accelerated motion. Find the angular acceleration of the grindstone and the linear acceleration of the cord.

10 lbs. = 320 poundals. Torque = 320 × 1.5 = 480 poundals feet.

I was found in the other problem to be 225 pounds feet squared.

Torque = $I\alpha$, $480 = 225\alpha$, from which $\alpha = 2.13$ radians per second per second.

Acceleration of the cord = $a = \alpha r = 2.13 \times 1.5 = 3.2$ ft. per second per second.

85. Acceleration of a Rolling Body.—When a body, such as a circular cylinder, rolls without slipping upon a plane surface, the points that are in contact with the surface are momentarily at rest, and the element of the cylindrical surface that is in contact with the plane is an instantaneous axis about which rotation takes place. Fig. 51 represents a solid circular cylinder resting upon a horizontal plane. Its mass is M. A cord is wound around it and a force F is applied horizontally to the cord. We wish to find an expression for the acceleration of the center of the cylinder. The moment of inertia about the axis is found by the equation proved in §78.

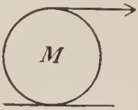


FIG. 51

$$I = I_o + Md^2,\ I = \frac{MR^2}{2} + MR^2 = \frac{3MR^2}{2}.$$

The torque is $2FR$, since the force is at a distance $2R$ from the axis.

$$2FR = I\alpha = \frac{3MR^2}{2}\alpha \text{ from which } \alpha = \frac{4F}{3MR}.$$

To find the acceleration of the string, $a = \alpha 2R = \dfrac{8F}{3M}.$

To find the acceleration of the center, $a = \alpha R = \dfrac{4F}{3M}.$

If the force had been applied at the center instead of at the top of the cylinder, its moment would have been half as great and the acceleration of the center would then have been $2F/3M$.

86. Bodies Rolling On an Inclined Plane.—When bodies of different shapes roll down an inclined plane, they will have different accelerations because their moments of inertia are different. The study of such cases becomes especially inter-

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

$$[illegible]$$

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[illegible]

[illegible]

88. Problem.—A solid cylinder of mass 10 lbs. and diameter 1 ft. rests upon a plane inclined at an angle of 30° with the horizontal. A string is wound around the cylinder and a force of 4 lbs. is applied to it in a direction parallel to the plane. Find the acceleration of the center of the cylinder.

Fig. 53 illustrates the problem. The component of the weight of the cylinder parallel to the plane is $Mg \sin 30^0 = 10 \times 32 \times \frac{1}{2} = 160$ poundals. This force is applied at the center of the cylinder, and its moment tending to produce counter-clockwise rotation about the axis in contact with the plane is $160 \times \frac{1}{2} = 80$ poundals feet. The moment of the force of 4 lbs. is in the opposite sense and is equal to $4 \times 32 \times 1 = 128$ poundals feet. This is greater than the other torque, and hence the cylinder will roll up the plane. To find the acceleration, equate the resultant torque to $I\alpha$:

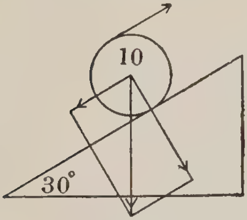


Fig. 53

$$128 - 80 = 48 = I\alpha = \frac{3}{2} MR^2\alpha$$

$$48 = \frac{3}{2} \times 10 \times \frac{1}{4} \times \alpha,$$

from which $\alpha = 12.8$ radians per second per second. $a = \alpha R = 12.8 \times \frac{1}{2} = 6.4$ ft. per second per second.

89. The Simple Pendulum.—A simple pendulum consists of a mass particle suspended by a weightless string. As a mathematical conception, this ideal pendulum has the advantage of having its mass all in one place, and there can be no question as to what is meant by the length of the pendulum. Thus the mathematical treatment is considerably simplified. A lead bullet suspended on a fine thread approaches the ideal pendulum so closely that results of the treatment of the ideal case find useful application. Our problem is to find the period of vibration of the simple pendulum expressed in terms of its length and the acceleration due to gravity. In Fig. 54, m is the mass particle and O is the point from which the pendulum is suspended. The vertical force of gravity, mg, may be divided into two components, one along the cord and the other at right angles to the cord. The first is transmitted through the cord to the point of support, and the other produces the motion of the pendulum. θ is the angle at any instant between the cord and the vertical. The component of the force that produces the motion is

$$mg \sin \theta,$$

m being the mass and $g \sin \theta$ the acceleration along the arc. The displacement of the pendulum bob from its mean position is the arc extending from m to the vertical line in the figure. x is the horizontal projection of the displacement and, for small amplitudes, it differs very little from the displacement itself. From the figure it is apparent that $\sin \theta = x/l$, l being the length of the pendulum. Substituting this value of $\sin \theta$ in the expression for the acceleration, we have

$$a = -\frac{gx}{l}.$$

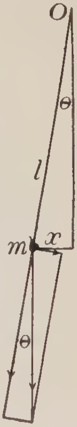


FIG. 54

The negative sign is used because x is measured to the left when a is toward the right. g and l are constants and x is variable as the pendulum swings; and if we set the displacement equal to x, we see that the acceleration is proportional to the displacement, and opposite in sign. This is the characteristic property of simple harmonic motion; hence, within the limits in which the arc may be regarded as equal to its horizontal projection, the motion is simple harmonic. Referring to §35, we recall that the acceleration in a case of simple harmonic motion is $\frac{-4\pi^2 x}{T^2}$. Equating this value of the acceleration to that deduced for the pendulum, we have

$$-\frac{gx}{l} = -\frac{4\pi^2 x}{T^2}.$$

Solving this equation for T,

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

From this equation it appears that the period of a simple pendulum varies as the square root of its length, and inversely as the square root of the acceleration due to gravity. Since x has disappeared from the equation, we may infer that the period is independent of the amplitude if the amplitude is small. It is the constant period that gives to the pendulum its value as a timekeeper in a clock.

When it is possible to measure the period and the length of the pendulum accurately, g is the only unknown remaining in the equation, and hence we have a convenient and accurate means of measuring the acceleration due to gravity.

90. Problems.—(1) If a simple pendulum 100 cms long has a period of 2 seconds, find the length of a pendulum whose period is 3 seconds. Since at any one place g is a constant, the only variables are T and l. The period is directly proportional to the square root of the length; hence

$$2 : 3 = \sqrt{100} : \sqrt{x}$$
$$\sqrt{x} = 15 \text{ and } x = 225 \text{ cm.}$$

(2) If the period of simple pendulum 100 cms long is 2 seconds, find the acceleration due to gravity.

$$T = 2\pi\sqrt{\frac{l}{g}},\ 2 = 2\pi\sqrt{\frac{100}{g}},\ \text{from which } g = \frac{987 \text{ cm}}{\text{sec.}^2}.$$

91. The Compound or Physical Pendulum.—In discussing the simple pendulum, a bullet suspended by a fine thread was mentioned as closely approximating the ideal pendulum. The length is usually measured from the point of suspension to the center of the bullet. In considering this as the length l appearing in the formula, a slight error is made; and the error is larger if a larger bullet is used. If the dimensions of the pendulum bob are large, the formula already derived fails to apply to the case, and we must resort to a new derivation that takes into account the fact that different parts of the swinging pendulum have different velocities and different accelerations.

Fig. 55 represents a compound pendulum suspended at the point S. The center of mass is at C at a distance from S equal to h. At a given instant when C is displaced from its mean position, as shown in the figure, the gravitational force Mg, acting at the centroid, has a moment about S equal to Mgx, x being the perpendicular distance between the center of suspension S and the vertical line through the centroid. Mgx is the torque that causes the acceleration of the pendulum, and since torque $= I\alpha$,

$$Mgx = I\alpha.$$

If a represents the acceleration of the centroid C,

$$a = \alpha h \text{ and } Mgx = \frac{Ia}{h}, \text{ from which } a = \frac{Mghx}{I}.$$

This is the absolute value of a. If we wish to express the fact that x is measured to the left when a is directed toward the right, and *vice versa*, we should write the value of a with a negative sign; then

$$a = \frac{-Mghx}{I}.$$

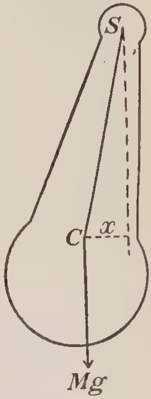


FIG. 55

If the amplitude of vibration is small, we may consider x as equal to the displacement of the centroid at any instant. M, g, h, and I are all constant, and x is the only variable in the right-hand member of the equation; hence the acceleration is proportional to the displacement and opposite in sign. The motion of the pendulum is, therefore, approximately simple harmonic motion, and the acceleration, from §35, is

$$a = -\frac{4\pi^2 x}{T^2}.$$

Equating these two values of the acceleration,

$$-\frac{Mghx}{I} = -\frac{4\pi^2 x}{T^2}.$$

Solving for *T*,

$$T = 2\pi\sqrt{\frac{I}{Mgh}}.$$

In order that the period of a compound pendulum may be computed, it is necessary to find first the position of the center of mass and second the moment of inertia about the axis of suspension. If the body has a regular geometric form, this may usually be done by computation; if the body is irregular in shape, it can only be done by experiment.

92. The Length of the Equivalent Simple Pendulum.—If a bullet on a fine thread is suspended beside a compound pendulum and the length of the thread adjusted till the two pendulums have the same period, the bullet and thread represent what is known as the equivalent simple pendulum. Since the periods are equal, we may write the expression for the period of a simple pendulum equal to that for the period of the compound pendulum; thus,

$$2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{I}{mgh}}.$$

Cancel the 2π and the *g* and square both members of the equation; then

$$l = \frac{I}{Mh}.$$

This is the length of the simple pendulum whose period is the same as that of the compound pendulum having a mass *M*, a moment of inertia *I*, and the distance from center of suspension to center of mass equal to *h*.

93. Application of Pendulum Formulæ to Special Problems.—(1) Equivalent length of simple pendulum for a long thin rod suspended from one end. In §77 the moment of inertia of a thin rod about its center of mass is found to be $\frac{ML^2}{12}$. To find the moment of inertia about an axis through one end,

$$I = I_\circ + Md^2, I = \frac{ML^2}{12} + \frac{ML^2}{4} = \frac{ML^2}{3}.$$

Substituting this in the equation $l = I/Mh$

$$l = \frac{ML^2}{3} \div \frac{ML}{2} = \frac{2}{3}L,$$

which shows that the equivalent simple pendulum is two-thirds as long as the thin rod.

(2) Circular disc, axis through the circumference perpendicular to the plane of the disc.

$$l = \frac{I}{Mh} = \frac{3/2\ MR^2}{Mh},$$

$$h = R;\ \text{hence}$$

$$l = \frac{3}{2}R = \frac{3}{4}D,\ D\ \text{being the diameter.}$$

(3) Thin ring or hoop, axis through the circumference perpendicular to the plane of the ring.

$$l = \frac{I}{Mh} = \frac{2MR^2}{MR} = 2R = D.$$

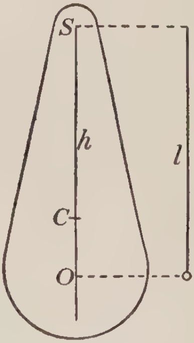


FIG. 56

This means that the simple pendulum whose length is equal to the diameter of the hoop will have the same period of oscillation as the hoop will have.

These three illustrations are chosen because the results here derived are easily checked by experiments on the lecture-table.

94. Center of Oscillation.—In Fig. 56 S is the center of suspension and C the center of mass of a compound pendulum. If we extend the line SC to a point O, making SO equal to the length of the equivalent simple pendulum, the point O is called the center of oscillation. The pendulum has the same period as if its entire mass were located at this point.

95. Center of Suspension and Center of Oscillation Interchangeable.—The diagram at the left in Fig. 57 represents a compound pendulum whose center of mass is C and whose center of oscillation is O. The equivalent simple pendulum is shown at the middle of the figure. At the right the same pendulum is shown suspended from what in its first position was the center of oscillation. We shall prove that the period of oscillation is the same for the pendulum suspended from O as when suspended from S. Let l be the length of the equivalent simple pendulum when the compound pendulum is suspended from S, and l' the length of the equivalent simple pendulum when the compound pendulum is suspended from O. If we can show that $l = l'$, we have proved the proposition.

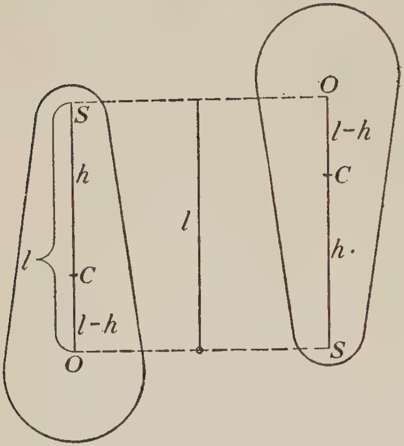


FIG. 57

$$I = I_o + Mh^2 \text{ and } l = \frac{I}{Mh} = \frac{I_o + Mh^2}{Mh}.$$

Clearing of fractions, $I_o + Mh^2 = Mhl.$

In the inverted position $I = I_o + M(l - h)^2$,

then

$$l' = \frac{I_o + M\,(l - h)^2}{M(l - h)} = \frac{I_o + Ml^2 - 2Mlh + Mh^2}{M(l - h)}.$$

For $I_o + Mh^2$ we may substitute Mhl from the equation above; then

$$l' = \frac{Mhl + Ml^2 - 2Mhl}{M(l - h)} = \frac{Ml^2 - Mhl}{M(l - h)} = \frac{Ml(l - h)}{M(l - h)} = l.$$

Thus we prove that the length of the equivalent simple pendulum is the same when the pendulum is suspended at O as when suspended at S, and hence it has the same period.

The facts thus established furnish an accurate means of determining l for a compound pendulum; and the period is expressed by the formula

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

A pendulum may be fitted with two adjustable knife edges to serve as axes, and their distance apart may be so adjusted that the pendulum has the same period when suspended at either edge. Then, if the positions of the knife edges are properly chosen, the length l is the distance between the knife edges and can be measured accurately. Such a pendulum has been used extensively by observers connected with the Coast and Geodetic Survey in their determinations of the acceleration due to gravity at different places throughout the world.

If the pendulum were a long thin uniform rod, there would be four points of suspension for which the period is the same. For example, if a meter stick hangs with its point of suspension at the zero end, the period would be the same if the center of suspension were at the $33\frac{1}{3}$ cm mark, the $66\frac{2}{3}$ cm mark, or the 100 cm mark. The distance l is $66\frac{2}{3}$ cm, not 100 cm or $33\frac{1}{3}$ cm.

96. Center of Percussion.—If a body, for example a meter stick, were suspended by a long cord from a high ceiling, as shown in Fig. 58, it would be comparatively free to move in a horizontal direction. Suppose a sharp blow were struck in a horizontal direction at the point O by means of a hammer. The stick would be set in motion, and its position after a very brief time might be represented by the dotted lines in the figure. It would be observed that the motion of the stick immediately after the blow is struck is a rotation about some fixed point, S. If the blow is applied rather near to the lower end of the stick, this point lies within the stick; while if it were applied below but near the centroid, the point S would be

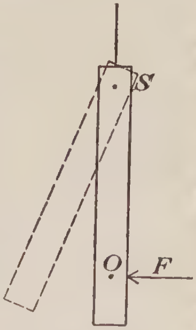


FIG. 58

far above the stick. If the conditions are such that the point S is within the stick, we might drill a hole through the stick at this point and suspend it upon a pin through the hole driven into the wall. If we repeat the blow, the axis of suspension experiences no jar or force at the moment the blow is struck. The point O where the force is applied is called the center of percussion corresponding to the particular axis through S. If the blow is applied elsewhere, S will have a different position. Those who play baseball know that there is a particular place in the bat where a swiftly moving ball may strike without causing a stinging sensation in the hands. The stick may be considered as a pendulum swinging about the axis through S. If, while the axis is fixed in position, we apply the blow at different places, we find that there is only one place at which it may be applied without a tendency to break the pin that supports it.

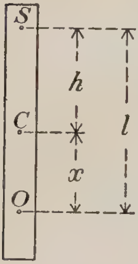


FIG. 59

It is our purpose to prove that the center of percussion is also the center of oscillation when the body is considered as a pendulum. Fig. 59 shows the same meter stick, and indicates the meaning of x and h. We represent the mass of the body by M, and assume that its centroid is at the point C. A horizontal force is applied at O. In §64 it was shown that we may resolve the effect of such a force into two parts, one a linear acceleration of the centroid equal to F/M, and the other an angular acceleration of the body about an axis through the centroid, given by the equation $Fx = I_o \alpha$, from which $\alpha = Fx/I_o$. As a result of the translation alone, S would move to the left; as a result of the rotation alone, it would move to the right. Since it remains stationary, the two accelerations must be equal and opposite. The first is $a = F/M$, the second is $a' = \alpha h$; and since $\alpha = \dfrac{Fx}{I_o}$, $a' = \dfrac{Fxh}{I_o}$. Putting $a = a'$, we have

$$\frac{F}{M} = \frac{Fxh}{I_o} \text{ and } x = \frac{I_o}{Mh}.$$

To find the distance l from S to O, add x and h; then

$$l = x + h = \frac{I_o}{Mh} + h = \frac{I_o + Mh^2}{Mh}.$$

But $I_o + Mh^2 = I$, the moment of inertia about an axis through S; hence

$$l = \frac{I}{Mh},$$

and this is exactly the expression for the length of an equivalent simple pendulum, which is also the distance from the center of suspension to the center of oscillation of the compound pendulum. Thus we prove that the center of percussion and the center of oscillation are the same point.

PROBLEMS

1. A meter stick weighs 200 grams. Find its moment of inertia about an axis at right angles to the stick and through a point 20 cm from one end. 346,64 [illegible]

2. If the stick of the preceding problem were made to rotate about the axis mentioned in the problem, making 5 revolutions per second, find its angular momentum.

3. A flywheel has a mass of 2,000 lbs. and a diameter of 12 ft. If, as a first approximation, we assume that all of its mass is in the rim, what is its moment of inertia? What energy is stored in it when it is making 90 revolutions per minute?

4. Assume that the wheel of the preceding problem is disconnected from the engine and mounted upon bearings that may be considered as frictionless. Suppose further that the wheel is at rest and that a rope is wound around it, then a force of 50 lbs. is applied to the rope; with what linear acceleration will the rope unwind?

5. A thin hollow cylinder weighing 96 lbs. is made to slide upon ice with a speed of 10 ft. per second without rotating. What is its kinetic energy? If the same cylinder were to *roll* with the same speed, what would its kinetic energy be?

6. A solid cylinder having a mass of 150 lbs. and radius of 2 ft. is mounted on a horizontal axis to rotate practically without friction. A cord is wound around it to which a force of 40 lbs. is applied. The cylinder is turned by the force; with what acceleration does the cord unwind?

7. The cylinder described in the preceding problem is placed on a horizontal surface, and the force of 40 lbs. is applied to its axis in a horizontal direction. There is sufficient friction to prevent slipping, but the resistance to rolling is negligible. Find the acceleration of the center of the cylinder. 5.6

8. If the force is applied horizontally to a string wound around the cylinder of the preceding problems, find the acceleration of the center of the cylinder and that of the string.

9. If, in problem 6, instead of a force of 40 lbs. applied to the string, there had been a mass of 40 lbs. suspended from the string, the result would have been different. Solve the problem and explain the difference.

10. Find the kinetic energy of a cannon ball whose mass is 160 lbs. if it is rolling on the floor with a speed of 10 ft. per second.

11. In the preceding problem, what part of the total kinetic energy is energy of translation?

12. If a solid homogeneous cylinder were put upon a plane inclined 30° to the horizontal, with what acceleration would it roll?

13. With what acceleration would a solid sphere roll upon a plane inclined 30° to the horizontal?

14. If a pendulum 100 cms long vibrates seconds, *i.e.*, has a period of two seconds, find the periods of pendulums 81, 64, and 49 cms long, respectively. How long is a pendulum whose period is .5 of a second?

15. At the surface of the moon a body would fall freely about 2 ft. 8 inches in the first second. How long would a pendulum be that would vibrate seconds there?

16. A hole is drilled through a meter stick at the 10-cm mark. A wire through the hole serves as an axis about which it vibrates as a pendulum. Find its period. $T = 2\pi\sqrt{\frac{I}{mgh}}$ T = 1.5

17. A cylindrical disk 2 ft. in diameter is suspended by a stiff wire 6 inches long, making the distance from the center of suspension to center of gravity 18 inches. Find the length of an equivalent simple pendulum. 1.83

18. Two pendulums of length, 100 cms and 169 cms, respectively, are started at the same instant in the same direction. How many vibrations will each make before they are together again and moving in the same direction? It is assumed that they have equal amplitudes.

19. A stiff wire, whose mass we shall neglect, is 75 cms long and is supported at one end so that it may vibrate as a pendulum. It carries a mass of 2 kilos 25 cms from the point of support, a mass of 4 kilos

2π

50 cm from the point of support, and a mass of 6 kilos 75 cm from the point of support. Find the length of the equivalent simple pendulum.

20. A thin circular hoop 40 cm in diameter hangs over a nail and vibrates in its own plane. Find the period of vibration.

64.28

$I_0 = \frac{MR^2}{2}$

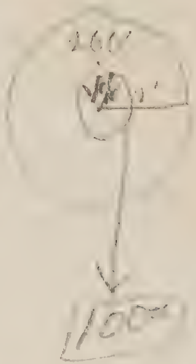
$I = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$

$\frac{3}{2}MR^2\frac{a}{R}$

$F = \frac{3}{2}Ma$

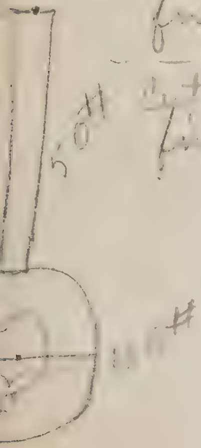
$F = \frac{3}{2}Ma$

moment of inertia



find period

cut circle 6" rod find period



(1) find the mass falls to rotate disc at 200/

(2) angular acceleration of disc and roller " of

(3) K.E. of rotation of disc in

(4) velocity of mass in 5 sec

(5) distance fall in 5 sec

CHAPTER VII

ELASTICITY

97. Stress, Strain, and Elastic Limit.—In discussing simple harmonic motion in §33, Hooke's Law was mentioned, and it was pointed out that when a body is distorted within certain limits the force with which it reacts is proportional to the distortion. This simple relationship makes possible many valuable computations in connection with the design of machinery and structures. For example, it is possible to compute in advance just how much a beam will be deflected or a piece of steel will be stretched by a given load. The forces involved in the reaction are called elastic forces, and the subject under consideration is called the theory of elasticity. We shall need clear definitions of two words that are frequently more or less confused.

Stress is the internal reaction against an externally applied load expressed as force per unit area. When the stress is the result of stretching a body, it is called *tension,* and when it is the result of an action that tends to compress the body, it is called *pressure.*

Strain is relative distortion, *i.e.,* the ratio of the distortion to the original magnitude whose distortion is measured. As an illustration of what this means, consider a wire of length L to be stretched by an amount l. The strain is the ratio l/L. If a body whose volume is V is put under pressure and its volume is decreased by an amount v, the strain is v/V. These two illustrations represent different types of strain.

Elastic Limit.—When a body is distorted so much that it fails to return to its initial dimensions and shape when the forces causing the distortion are reduced to zero, the body is said to have passed its elastic limit. The elastic limit is expressed as a limiting stress, and is that stress beyond which the body experiences a permanent distortion. For steel the elastic limit may be more than 50,000 pounds per square inch.

Hooke's Law may now be restated. When applied forces distort a body within certain limits, the stress set up is proportional to the strain produced.

98. Coefficients of Elasticity.—Within the limits in which stress is proportional to strain, the constant ratio of stress to strain is called a coefficient of elasticity. Since there are several different types of strain, such as change of length per unit length and change of volume per unit volume, it is evident that several different coefficients of elasticity must be defined.

99. Young's Modulus.—If only the change of length is measured, the ratio of stress to strain is called Young's modulus. When the specimen under consideration is a long wire, two equal forces are applied in opposite directions to its two ends. Calling one of these forces F, the stress is equal to F/a, a being the area of cross-section of the wire, which is considered as uniform throughout its length. If the initial length is L and the change in length is l, the strain is l/L, and

$$\text{Young's modulus} = \frac{\text{Stress}}{\text{Strain}} = \frac{F}{a} \div \frac{l}{L} = \frac{FL}{al}.$$

It is easy to measure F and L because they are large; but the accuracy of a determination of Young's modulus depends upon how accurately a and l can be measured. This is a laboratory problem and is treated in the laboratory course. Young's modulus is measured in dynes per square cm or in poundals per square inch or in pounds per square inch. Its value for steel is in the neighborhood of 30,000,000 pounds per square inch. Young's modulus is used more than any other coefficient of elasticity in practical work.

It should be observed that, while Young's modulus is determined from measurements made upon a particular piece of the substance, it is really a property of the material of which the specimen is made. The result should be the same irrespective of the dimensions of the specimen measured, so long as the different specimens are all of the same substance. This is also true of the other elastic coefficients to be mentioned in succeeding sections. The elastic coefficients of a large number of substances are known; hence it is usually possible to compute the strain resulting from a given load on the specimen.

100. Coefficient of Volume Elasticity.—A change of volume results from stretching a wire, but usually the change of length is all that is of particular interest and is all that is measured in determining Young's modulus. When the volume of a body is decreased by pressure applied to it, the pressure is equal and opposite to the stress, and the strain is the change of volume per unit-volume or $\frac{v}{V}$. $p \div \frac{v}{V} = \frac{pV}{v}$ is the coefficient of volume elasticity. This is also expressed in dynes per cm^2 or pounds per inch^2.

101. Coefficient of Simple Rigidity.—A third type of strain, called a shearing strain, measures a change of shape rather than a change of volume. In Fig. 60 a block in the form of a rectangular parallelopiped is shown. It is glued or screwed to a table top, and a board ab, covering the entire top, is fastened to it. A force F directed toward the right is applied to the board in a direction parallel to the surface of the table. The table exerts an equal reaction upon the block in the opposite direction. The effect of this pair of forces is to distort the block and give it the shape indicated by the dotted lines. The stress set up in the block is measured by the applied force divided by the area of the top of the block. This stress is called a shearing stress. The strain produced by the action of the pair of forces is the displacement of the upper surface of the block divided by its height. This is l/h in the figure, and is evidently the measure of the angle ϕ. Such a strain

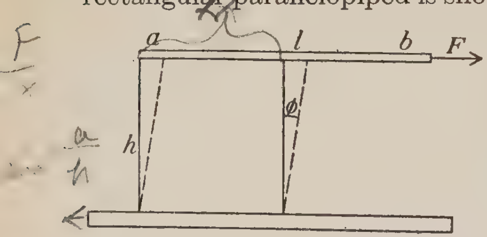


FIG. 60

is called a shearing strain, or simply the shear. The ratio of shearing stress to shearing strain is known as the coefficient of simple rigidity. It is equal to $\frac{F}{a} \div \frac{l}{h} = \frac{Fh}{al}$.

When a shaft is twisted it experiences a shearing strain, and it is the coefficient of simple rigidity that is used in computing the effect produced upon it by a torque.

102. Stresses in a Beam.—If a beam supported at its two ends carries either a distributed load or a load concentrated at the middle, as shown in Fig. 61, there are three different stresses to be considered. The beam bends under the load, causing a horizontal compression of the upper part and a stretching of the lower part of the beam. In order to compute the effects of the stresses in these two parts, Young's modulus for the material of which the beam is composed must be known. A shearing stress is produced as a result of the downward force of the load and the upward forces at the points of support. This stress is zero at the middle and increases toward the two ends of the beam. When Young's modulus for the material is known and the dimensions of the beam are given, it is possible to compute the deflection of the beam for any given load. It is also possible to compute the load that may be safely placed on the beam if the maximum safe stress has been determined.

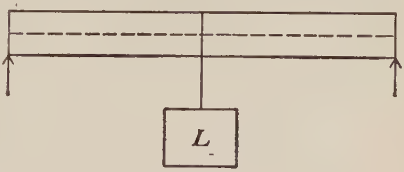


FIG. 61

YOUNG'S MODULUS

	Dynes per cm²	*Lbs. per inch²*
Aluminum	7×10^{11}	9.5×10^6
Brass	$(9.5 \text{ to } 11.2)\ 10^{11}$	$(13 \text{ to } 15)\ 10^6$
Copper	$(11.7 \text{ to } 12.3)\ 10^{11}$	$(16.5 \text{ to } 17.5)\ 10^6$
Cast iron	13.5×10^{11}	18.5×10^6
Wrought iron	19.6×10^{11}	29×10^6
Steel	$(21 \text{ to } 24.5)\ 10^{11}$	$(29 \text{ to } 32)\ 10^6$

Elasticity = $\frac{\text{stress}}{\text{strain}}$

453.6 × 98 [illegible]

PROBLEMS

32 ×

1. If Young's modulus for steel is taken as 32,000,000 lbs. per square inch, find what it would be, expressed in dynes per square cm.

2. A steel wire having a cross-sectional area of .01 square inch is 20 ft. long. Find the stretch produced by a load of 250 lbs.

3. A steel wire is twice as long as a copper wire. It is found that a given load stretches the steel wire three times as much as it does the copper wire. Find approximately the ratio of their diameters.

4. A vertical steel cable 100 ft. long is stretched 1.5 inches by a force of 2,500 lbs. How much work is done in stretching the wire? Observe that the force varies from zero to a maximum as the length increases. Show that the work varies as the square of the force.

5. Assume the safe working stress of steel cable to be 18,000 lbs. per square inch. A certain elevator car with its maximum load weighs 3,000 lbs. Its maximum acceleration is 8 ft. per second per second. What cross-section should the cable have?

6. Two vertical parallel wires are of the same size and length. They are fastened together at both ends and are used to support a heavy weight. If one is of brass and the other of steel, what part of the weight will each carry.

7. Two tuning forks are of exactly the same size and shape, one of brass, the other of steel. Which will have the longer period of vibration? Take the density of steel as 7.8 grams per c.c. and the density of brass as 8.4 grams per c.c. and find the ratio of the periods. (Refer to the laws of simple harmonic motion.)

8. The elastic limit for steel wire is 60,000 lbs. per square inch. How much will a steel wire 20 ft. long stretch without permanent distortion?

9. A copper wire is stretched between two poles for electric lighting purposes, under such conditions that it is stretched to its elastic limit, *i.e.*, 10,000 lbs. per square inch. Show that a steel wire whose diameter is .45 that of the copper wire would, under the same conditions, support its own weight and in addition that of the copper wire without permanent stretch. 17,000 [illegible]

10. The breaking load for copper wire is about 4,000,000 grams per sq. cm; its density is about 8.9 grams per c.c. If a copper wire were suspended vertically from one end, how long would it have to be to break because of its own weight?

$8.9 \pi r^2 l = 4,000,000 \times \pi r^2$

$Y_s = \frac{F_s l}{a_s e}$

$Y_B = \frac{F_B l}{a_B e}$

$= 32 \times 10^5$

$Y_{cu} = \frac{F l_{cu}}{\frac{1}{4}\pi d^2 e}$

$W = FS$

$1250 \frac{1.5}{12}$

$W = F\frac{S}{2}$

$W = \frac{F^2}{2C}$

$W = F \cdot l$

$Y_m = \frac{FL}{Al}$

$l = \frac{FL}{Y_m A}$

$= \frac{F^2 L}{Y_m A}$

MECHANICS
OF FLUIDS

$F = ma$

(5) $T - 3000g$

$T = 375{,}000$

$\frac{37500}{180{,}000} = \frac{5}{24}$

$Y_m = \frac{FL}{Al}$

$Y_s = \frac{F_s l}{a\, l_s}$

$\frac{Y_s}{Y_B}$

$\frac{320}{150}$

(9)

$R_s l = R X$

$= 175$

$T =$

CHAPTER VIII

HYDROSTATICS

103. Definitions.—The density of a substance is its mass per unit volume. The unit of density is one gram per cubic cm or one pound per cubic foot. A fluid is a body whose coefficient of simple rigidity is zero. This is equivalent to saying that when a fluid suffers a change of shape it has no tendency to regain its former shape. It may offer a resistance to change of shape due to its inertia and also a resistance due to an internal friction called viscosity; but these forces exist only while there is motion; they are opposed to the motion and hence cannot operate to restore the initial form of the body. A fluid does resist any attempt to decrease its volume; after compression it will return to its original volume if the pressure and temperature are restored to their initial values. This is true in all cases, irrespective of the magnitude of the change; hence, for changes of volume, a fluid has no elastic limit.

Fluids are classified as liquids and gases. A body of liquid has a definite volume which is not determined by the volume of the vessel containing it, and when it is placed in an open vessel it shows a free surface. The volume of a gas, on the contrary, is determined by the volume of the vessel containing it. The gas fills the available space and shows no free surface. Liquids usually have such large coefficients of volume elasticity that their changes of volume, due to ordinary changes of pressure, go unnoticed; and it is customary in mechanical problems to treat liquids as if they were incompressible. Gases, on the contrary, are easily compressible and their changes of volume resulting from small changes of pressure are easily observed.

104. Pressure at a Point.—In this and several succeeding topics, the discussion relates to fluids at rest. A fluid exerts a force upon the walls of the vessel containing it and upon the surface of any body immersed in it. In an open vessel the force is due to the weight of the fluid and to the weight of the air above it. In a closed vessel the force may be in part an elastic reaction resulting from compressing the fluid. The pressure on an immersed surface is the force per unit area exerted upon the surface. Pressure is measured in dynes per square cm or in pounds per square inch. The pressure is uniform if the surface is horizontal; otherwise it varies from point to point upon the surface. The pressure at a point in a fluid is measured by the force per unit area that would be exerted at the point upon a surface containing the point. The pressure at a point in a fluid is the same in all directions. This is necessarily true, for if we consider a small element of the fluid, it is clear that it could not maintain its shape, if, for example, the vertical pressure were greater than the horizontal pressure. A solid could maintain these unequal stresses because of its rigidity, but a liquid does not possess this property. Since the fluid is assumed to be at rest, there is no direction in which the pressure is either greater or less than in other directions.

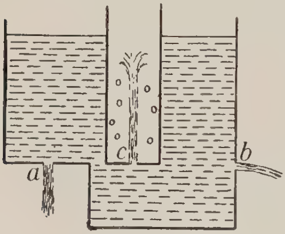


Fig. 62

Suppose that a tank were built, as shown in Fig. 62, with an orifice at *a* and another at *b*. If the tank were filled with water, the water would flow from the two orifices with the same speed. The speed of flow is determined by the pressure, and since the speeds are equal, the downward pressure at *a* is equal to the horizontal pressure at *b*. A hollow cylinder, closed at one end, may be thrust downward into the water, and if a hole *c* is made in the closed end, the water will run into the cylinder. When the orifice *c* is at the same depth as *a* and *b*, the water will flow upward through *c* with the same speed as it flows downward through *a* or horizontally through *b*. In fact, if the cylinder were turned so that the jet coming into it were to take any other direction than the vertical, it would be found that the velocity of the jet is the same, provided the depth of the orifice below the surface of the water remains the same. This illustrates the fact that the pressure is the same at all points in a horizontal plane, and also that the pressure at a point is the same in all directions.

105. Force Normal to the Surface.—The direction of the force exerted by a fluid upon an element of surface is perpendicular to the surface. Consider a vessel of the shape illustrated in Fig. 63. If the force on an element at *d* of the surface *ab* were not perpendicular to the surface, it would have some other direction, as *cd*. The reaction of the surface against the fluid, being equal and opposite to the force due to the fluid, would then be represented by the vector *dc*. The reaction *dc* may be divided into two components, *df* parallel to the surface, and *de* perpendicular to it. The component *df*, being tangential to the surface, would produce the same kind of effect as the force *F* in Fig. 60, *i.e.*, it would cause a shearing stress in the fluid. But, from the definition of a fluid, there can be no shearing stress; hence it follows that the force can have no component parallel to the surface upon which it acts and therefore must be perpendicular to it.

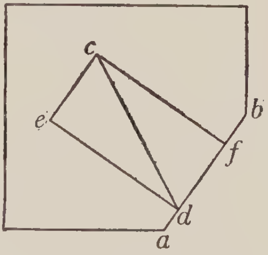


FIG. 63

106. Work Done Upon a Fluid.—A piston B, shown in Fig. 64, is arranged to slide in a cylinder with so little friction that frictional forces may be neglected. The cylinder and the system to which it is connected by the pipe at the right are filled with water or some other fluid. Assume that the variation of pressure over the surface of the piston is negligible and that the pressure remains constant as the piston is moved.

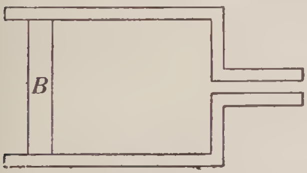


FIG. 64

$P = \frac{F}{A}$

$P = K_1 \alpha g$

Let P be the pressure and A the area of the piston. The force exerted upon the piston by the fluid is PA; and if the piston is pushed inward a distance L, the work done on the fluid is PAL. If P is in dynes per cm², A in cm², and L in cm, the work is expressed in ergs. It is apparent that the product AL is the volume of that portion of the cylinder traversed by the piston, and is therefore equal to the volume of the fluid pumped through the pipe into the system beyond. Substituting V for this volume, the work done equals PV. This offers a convenient way of computing the work done when water is pumped into a system of pipes in which the pressure is known.

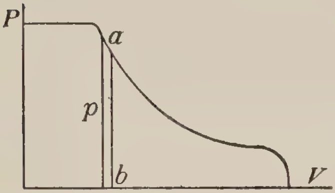


FIG. 65

If the piston is pushed outward by the force exerted by the fluids, the fluid does the work. In the hydraulic elevator, water from the city mains exerts a force upon the piston and does the work of operating the elevator. In this case also the product PV is the measure of the work done. In computing the work done by a steam engine the same principle is used, but the problem is complicated by the fact that the pressure is not constant. In this case it is customary to compute the work from an indicator diagram. A piece of apparatus known as an indicator is used in automatically plotting a curve which shows the steam pressure in the cylinder at every position of the piston. The ordinate of the curve represents the steam pressure, and the abscissa is proportional to the distance of the piston from one end of the cylinder. The curve traced during the expansion usually has the form shown in Fig. 65. The volume of a cylinder is proportional to its length; hence the abscissæ are proportional to the changes in volume occurring since the beginning of the stroke. The ordinates are proportional to the pressures for the different positions of the piston. The area of a small strip, such as ab, is equal to the product of the ordinate, *i.e.*, the pressure, and a small portion of the abscissa, *i.e.*, a small change of volume. This product is equal to the work done on the piston during that part of its motion represented by the width of the strip, provided the ordinates are measured in units of pressure and the abscissæ in units of volume. The work done during the entire stroke is measured by the sum of all such areas as *ab*, *i.e.*, by the area under the curve.

At the end of the stroke the exhaust valve is opened so that the pressure is reduced to a low value, and as the piston returns to its initial position the pressures are represented by the ordinates of the lower curve in Fig. 66. Some work is done in expelling the exhaust steam from the cylinder, and this work is represented by the area under the lower curve. This is work done by the engine upon the steam, rather than by the steam upon the engine; hence it should be deducted from the area under the upper curve to find the resultant work done by the engine. Thus we see that the resultant work during the stroke is represented by the area included within the closed curve. A similar diagram is taken to show the work done by the steam that enters the cylinder at the opposite end and presses upon the opposite side of the piston. The sum of the two areas represents the work done for each revolution of the flywheel, and this sum multiplied by the number of revolutions per minute determines the work done per minute.

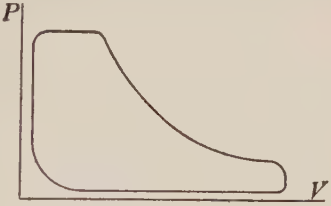


Fig. 66

107. Pascal's Law and the Hydraulic Press.—A diagram of a hydraulic press is shown in Fig. 67. A large cylinder is fitted with a piston which carries the load. This cylinder is connected by a small pipe to a force pump which supplies it with water. When the pistons are at rest and the valve between the cylinders is open, the water filling the system may be considered as a single body of fluid, and Pascal's Law may be applied in finding the mechanical advantages of the machine. Pascal's Law states that "a force exerted upon any area of an enclosed fluid is transmitted undiminished to every equal area of the surface enclosing the fluid." This does not say that the pressure is everywhere the same within the enclosure. This latter statement would be true only if the fluid were without weight. In a machine like the hydraulic press, the pressure applied at the pump is so great that the pressure due to the weight of the fluid is insignificant by comparison, and may be neglected, as will be done in the remaining portion of this section.

Let P be the value of the pressure, A the area of the cross-section of the large piston, and a the area of cross-section of the piston in the pump. Then PA is the force exerted by the water upon the large piston, and Pa is the force upon the small piston. Since action and reaction are equal, Pa also represents the downward force exerted by the piston upon the fluid. Disregarding the weights of the pistons, PA represents the load on the machine, and Pa the force applied to the machine. The mechanical advantage of the hydraulic part of the machine is then the ratio of these two forces, *i.e.*, PA/Pa. Canceling P and observing that the areas of circles are to one another as the squares of their diameters, we have

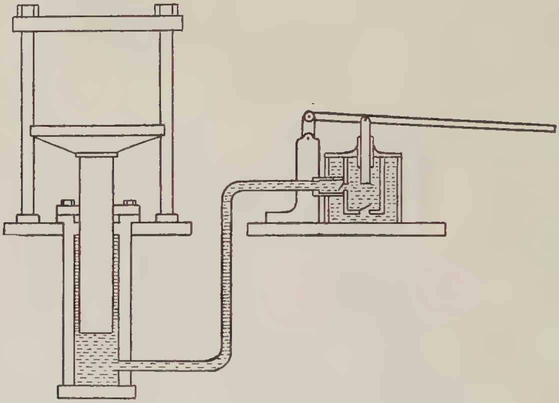


Fig. 67

$$\text{mechanical advantage} = \frac{A}{a} = \frac{D^2}{d^2}.$$

If the lever arms of the pump handle are L and l respectively, the mechanical advantage of the lever is L/l; and for the machine as a whole,

$$\text{mechanical advantage} = \frac{D^2L}{d^2l}.$$

By making the press cylinder large and the pump cylinder small the mechanical advantage may be made very large.

108. Pressure Due to the Weight of a Liquid.—In Fig. 68, a tank is represented having a horizontal bottom. The tank is partially filled with a liquid which we may regard as practically incompressible. Let ab represent a vertical column of the liquid having a square cross-section one cm on a side. The pressures on the sides of this column are horizontal, and the forces on opposite sides are equal and opposite; hence their resultant is zero. The weight of the column must then be supported by the bottom of the tank; the pressure on the bottom is equal to the weight of a column of the liquid one square cm in cross-section and having a height equal to the depth of the liquid. Let h be the depth of the liquid, and d its density or mass per unit volume. There are h cubic cm of the liquid resting upon each cm^2 of the bottom of the tank, and the mass of this liquid is hd grams. Its weight expressed in absolute units is hdg dynes. This expression represents the pressure, since it is the force per unit area exerted upon the bottom of the tank. The pressure at a point in the liquid is the same in all directions; therefore the expression hdg represents the pressure in any direction at a point h cm below the surface of the liquid. If English units were used, h would be measured in feet, d in pounds per cubic foot, and g would be approximately 32 ft. per second per second. The pressure, hdg, would then be expressed in poundals per square foot. If the factor g is omitted, the pressure is represented numerically as hd pounds per square foot.

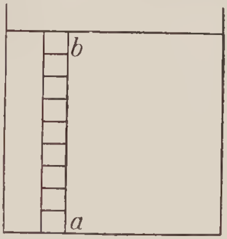


Fig. 68

109. Resultant Force on an Immersed Surface.—A plane surface whose area is A is immersed in a liquid of density d. By the preceding section, the pressure at any point of the surface is hdg, and it is perpendicular to the surface. Imagine the surface to be divided into a large number of very small parts, having areas a_1, a_2, a_3, etc. The distances of these small surfaces below the surface of the liquid are h_1, h_2, h_3, etc., respectively. The pressure may vary widely over the large surface, being greater at points near the lower edge than at points farther up. If, however, the areas a_1, a_2, a_3, etc., are taken sufficiently small, the pressure over any one of them may be considered as uniform. The uniform pressure upon a surface multiplied by its area gives the force acting upon the surface. The force acting upon the surface whose area is a_1 is then a_1h_1dg, and similar expressions represent the forces acting upon the other surfaces. Since these are parallel forces, their resultant is equal to their sum, and

$$F = a_1h_1dg + a_2h_2dg + a_3h_3dg + \text{etc.}$$

d and g are factors occurring in every term of the series; hence

$$F = dg(a_1h_1 + a_2h_2 + a_3h_3 + \text{etc.}) = dg\Sigma ah.$$

We may find an expression for the distance of the center of area of a plane surface from a given plane of reference by a method similar to that used in §62 for finding the center of mass of a body. If a represents a small element of area and h its distance from the plane of reference, the distance of the center of area from the plane of reference is H in the following equation:

$$H = \frac{\Sigma ah}{\Sigma a}.$$

In our present problem, the plane of reference is taken as the free surface of the liquid, and H means the distance of the center of the area pressed upon below the surface of the liquid. Clearing the last equation of fractions and substituting A for Σa, we have $\Sigma ah = AH$. If AH is substituted for Σah in the expression for the resultant force, the equation becomes

$$F = AHdg.$$

This is the magnitude of the resultant force in absolute units. Its point of application will be discussed in the next topic. If it is desired to express the force in gravitational units, the factor g is omitted. If the immersed surface were a curved surface, the forces acting upon its different parts would not be parallel forces. The methods employed in this paragraph would not apply to this case, and the expression $AHdg$ would not represent the resultant force.

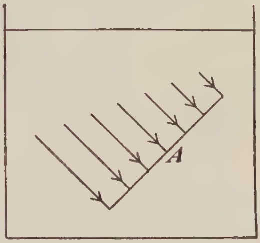


FIG. 69

110. Center of Pressure.—The point of application of the resultant of the hydrostatic forces acting upon an immersed surface is called the center of pressure. In determining the position of the center of pressure, we shall confine our attention to two special cases. In Fig. 69 the arrows indicate the direction of the pressures, and the length of the vectors represent roughly the magnitude of the pressures at different points on the surface. If we imagine the surface to be turned toward a horizontal position, all the vectors will approach the same magnitude and will be equal when the surface is horizontal. The resultant force is then applied at the center of area. This is the first special case mentioned above. The second is that of a plane surface in the form of a rectangle with one of its edges in the surface of the liquid. The pressure is zero at the upper edge and has a maximum value at the lower edge. Since the pressure is proportional to the depth, it varies uniformly from top to bottom of the surface. The vectors in Fig. 70 represent the pressures at various points on the surface AB. If the surface were divided into small parts having equal areas, the vectors of Fig. 70 might be used to represent the forces acting upon these surfaces. To find the point of application of the resultant of these parallel forces, consider another problem

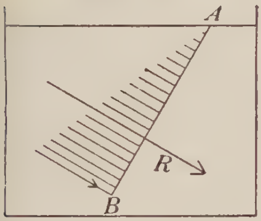


FIG. 70

that is closely analogous to this one. Fig. 71 represents a right triangle cut from thin sheet metal and divided into a large number of strips by equidistant lines parallel to the short side. The gravitational forces on the strips may be represented by vectors whose lengths are proportional to the areas of the strips. The forces will vary uniformly from zero to a maximum, just as the forces do in Fig. 70. Now we know from geometry that the center of gravity of the triangle is at the intersection of the medians, *i.e.*, two-thirds of the way from a vertex to the opposite side measured along the median. Since the forces represented in Fig. 70 are distributed in the same way as those in Fig. 71, it follows that the resultant hydrostatic force will cut the surface *AB* at a point two-thirds of the way from *A* to *B*. This point must not be confused with the center of area which, for the rectangle, is half way between the upper and lower edges.

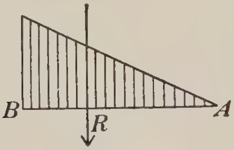


FIG. 71

111. Moments Due to Hydrostatic Forces.—This topic is devoted to an application of principles studied in earlier sections. No part of it need be memorized; yet the student should aim to understand the matter so that he can reproduce what is here given and apply the principles to other problems.

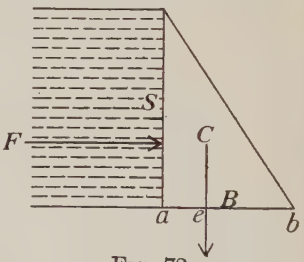


FIG. 72

Among other quantities related to the design and construction of a dam, the moment tending to overturn the dam must be determined, and also the moment of the gravitational force tending to maintain it in its upright position. Fig. 72 represents the cross-section of a dam with the water pressing on the left side. Let *S* be the depth of the water, and *B* the thickness of the dam at the base. The length of the dam is perpendicular to the plane of the diagram and is represented by *L*. The area pressed upon by the water is *LS*. In the equation $F = AHdg$, $H = S/2$. d for water is 62.4 lbs. per cu. ft. Then $F = LS \times \frac{S}{2} \times 62.4\,g$ poundals. The point of application of the force is at the center of pressure, *i.e.*, at a distance $S/3$ from the lower edge of the surface. If the dam were to turn over without breaking as a result of hydrostatic forces, it would have to turn around the edge at b; hence we find the moment of the force F about b.

$$\text{Moment of } F = LS \times \frac{S}{2} \times 62.4 \times g \times \frac{S}{3} \text{ poundals feet.}$$

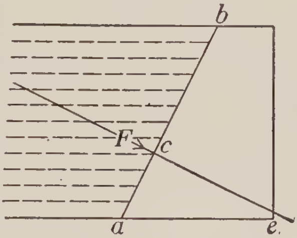


Fig. 73

For simplicity the section of the dam is here shown as a triangle. The center of mass is at the intersection of the medians of the triangle, and the resultant gravitational force acting upon the dam may be represented by a vector which intersects the base at e, so placed that eb is two-thirds of the length B of the base. The volume of the dam is $\frac{LBS}{2}$ and its mass is $\frac{LBSD}{2}$, D being the density of the material of which the dam is made. Its weight is $\frac{LBSDg}{2}$ poundals, and the moment of this force about the edge b is

$$L \times B \times \frac{S}{2} \times D \times g \times \frac{2B}{3} \text{ poundals feet.}$$

In order that the dam may be a stable structure, this moment must be greater than the moment tending to overturn it. The problem might be stated in this way: Find the least density which the material composing a dam of given dimensions may have if the dam is to be a stable structure. In this case we put the two moments equal and solve the equation for D.

Fig. 73 illustrates a type of construction having some obvious advantages. The side *ab* against which the water presses is inclined to the vertical. The vector F represents the resultant force exerted by the water. Its point of application is at *c*, so situated that *ac* equals one-third of *ab*. The perpendicular distance from the line of the force to the point *e* is smaller than if the force were horizontal, and thus the overturning moment is made smaller than in the case illustrated by Fig. 72. By turning the surface of the dam a little farther from the vertical, it is possible to make the resultant force pass through the lower corner of the dam and thus reduce the overturning moment to zero.

112. Problem.—A dam has a rectangular surface 40 ft. wide and 12 ft. high and lying in a vertical plane. Two horizontal lines are drawn upon it dividing the surface into three equal parts. Find the resultant force acting upon each of the three parts, assuming that the water comes to the top of the dam. Also find the moment tending to overturn it. Refer to Fig. 74.

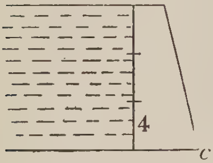


FIG. 74

The force expressed in pounds is numerically equal to AHd; hence

$$F_1 = 4 \times 40 \times 2 \times 62.4 \text{ pounds.}$$
$$F_2 = 4 \times 40 \times 6 \times 62.4 \text{ pounds.}$$
$$F_3 = 4 \times 40 \times 10 \times 62.4 \text{ pounds.}$$
$$\text{Total Force} = F = 12 \times 40 \times 6 \times 62.4 \text{ pounds.}$$

Perpendicular distance from force to axis *c* is 4 ft.

Moment of the resultant force is M.

$$M = 12 \times 40 \times 6 \times 62.4 \times 4 \text{ pounds feet.}$$

113. The Principle of Archimedes.—Imagine a cubical block 2 ft. on each edge immersed in water, with its upper face 2 ft. below the surface of the water, as illustrated in Fig. 75. The horizontal forces on opposite sides of the block are equal and opposite; hence their resultant is zero. The downward force on the upper surface is 4 × 2 × 62.4 pounds, and the upward force on the lower surface is 4 × 4 × 62.4 pounds. The upward force is greater than the downward force by an amount equal to the weight of 8 cubic ft. of water. The volume of the water displaced by the block is 8 cubic feet. Hence it is apparent in this particular case that the block is buoyed up by a force equal to the weight of the fluid displaced.

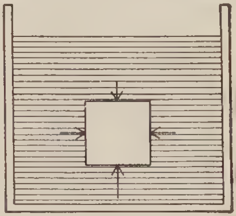


FIG. 75

To deal with the general case, suppose a portion of a fluid, either liquid or gas, to be solidified without change of density. This portion may have any shape whatsoever. Assuming that the fluid was at rest before the solidification took place, it would still be at rest afterward. This could be true only if the resultant of all the forces acting upon the solid portion were zero. Then the buoyant force due to fluid pressure must be equal to the weight of the solid. Since it was assumed that the density of the solid portion is equal to that of the fluid, it follows that the buoyant force is equal to the weight of the fluid displaced. Obviously the buoyant force would be the same if the fluid were displaced by any body other than the one discussed above, provided it had the same size and shape; so the proposition is perfectly general, and any body immersed in a fluid is buoyed up by a force equal to the weight of the fluid displaced. This fact was first announced by Archimedes more than 2,000 years ago. It is known as the *Principle of Archimedes.*

BF = [illegible]
= Vd

To find the point of application of the resultant buoyant force, consider again the solidified portion of the fluid. Since it is in equilibrium under the action of the resultant buoyant force and the resultant gravitational force, these two forces must lie in the same straight line. But the resultant gravitational force is applied at the center of gravity of the solid; hence the resultant buoyant force must lie *in the vertical line passing through the center of gravity* of the displaced fluid.

114. Specific Gravity.—The specific gravity of a body is the ratio of its weight to the weight of an equal volume of some substance which is adopted as a standard. The substance that is usually taken as a standard for solids and liquids is water at the temperature 4° centigrade. For gases, either hydrogen or air at standard conditions of temperature and pressure is taken as a standard. It is apparent that specific gravity, being a ratio of two weights, is an abstract number, and its value for a particular body is independent of the units in terms of which the weights are expressed.

115. Specific Gravity of a Solid.—If the body sinks when put into water, its specific gravity may be found as follows: Let W be the weight of the body in air. The buoyant force due to the air is neglected. Let W' be the apparent weight of the body when immersed in water. $W - W'$ equals the buoyant force due to the water. By the Principle of Archimedes, $W - W'$ is also the weight of the water displaced. The volume of the water displaced is equal to the volume of the body displacing it; then, by the definition of specific gravity,

$$\frac{W}{W - W'} = \text{specific gravity of the solid.}$$

To find the specific gravity of a body that floats in water, a sinker is fastened to the body to keep it immersed. The sinker is first tied to a scale pan by a thread. It is then immersed in water and shot or sand put in the other pan to cause a balance. Then the body whose specific gravity is to be found is put into the pan to which the sinker is attached and weighed. Let W be its weight. The body is then transferred from the pan to the water and tied to the sinker without making any change in the weights. Since the body tends to float, an additional weight W' is necessary in the pan carrying the sinker to restore the balance. The buoyant force is then W' and this is the weight of the water displaced according to Archimedes' Principle. Thus the specific gravity is found to be

$$\frac{W}{W'}.$$

116. Specific Gravity of a Liquid.—Several methods of measuring the specific gravity of a liquid suggest themselves. Making use of the Principle of Archimedes, the following process may be used: A solid which will sink in the liquid and which does not dissolve in it is weighed first in air, then in water, and finally in the liquid whose specific gravity is to be determined. Let W be the weight of the solid body in air, W' the weight in water, and W'' the weight in the liquid. $W - W''$ equals the weight of the liquid displaced, and $W - W'$ equals the weight of the water displaced; then

$$\text{the specific gravity} = \frac{W - W''}{W - W'}.$$

Another method makes use of a specific gravity flask, which is a small bottle with a glass stopper ground to fit the neck of the bottle. The stopper has a capillary hole through it so that the excess liquid may get out as the stopper is pushed into the neck of the bottle till it fits tightly. When the stopper is in place, the volume of liquid in the bottle is definite. The flask is first weighed when it is empty, then when filled with water, and finally when filled with the liquid whose specific gravity is to be determined. The ratio of the weight of the liquid filling the flask to the weight of the water that fills it is the result sought.

117. Density and Specific Gravity.—The density of a substance is defined as the mass per unit volume of the substance. The density of water at 4° centigrade is very nearly one gram per cubic centimeter. The specific gravity of water is taken as unity, and if the specific gravity of some other substance—iron, for example—is 7.6, a certain volume of iron is 7.6 times as heavy as an equal volume of water. Its density is then 7.6 grams per cubic cm. Thus it appears that the numerical value of the density of a substance expressed in grams per cubic cm is equal to its specific gravity. In the English system of units, this relation does not exist. The specific gravity of water is unity, but its density is 62.4 pounds per cubic foot. The specific gravity of iron is about 7.6, but its density is about 480 pounds per cubic foot.

118. Comparison of Densities of Liquids.—A branched tube of glass is so arranged that some of the air can be pumped from it at the top. The two branches dip into cups containing different liquids, as shown in Fig. 76. When the air pressure in the tubes is reduced, the liquids rise in the tubes because of the action of air pressure on the outside. If the two liquids have different densities, the heights of the two columns will be different, but the pressures supporting them will be equal. h and h' represent the two heights of the columns, and d and d' the densities of the liquids; then

$$hdg = h'd'g,$$

from which

$$\frac{d}{d'} = \frac{h'}{h},$$

i.e., the densities are inversely as the heights of the columns. If water is used in one of the cups, and if h' is the height of the water column, the ratio h'/h measures the specific gravity of the other liquid. If the tubes are rather long, measurements of h and h' can be made with considerable accuracy. Observations can be secured very quickly if the apparatus is kept set up ready for use.

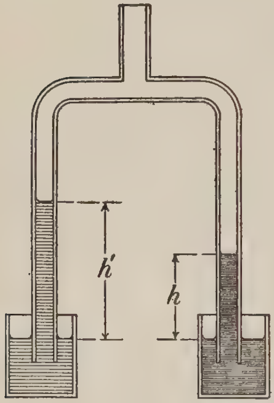


Fig. 76

119. Floating Bodies.—When a body floats at the surface of a liquid, the buoyant force is equal to its weight; hence the weight of the liquid displaced must equal the weight of the body. When the statement is made that a certain ship has a displacement of 10,000 tons, it is merely another way of saying that the weight of the ship and its contents is 10,000 tons. We have seen that the point of application of the resultant buoyant force is the center of gravity of the displaced fluid. This point is called the center of buoyancy and is of special interest in connection with the design of boats. Fig. 77 represents the cross-section of a boat. B is the center of gravity of the displaced water, and C is the center of gravity of the boat. The center of gravity of the boat is almost always considerably above the center of buoyancy. If the center of mass of a pendulum were vertically above the axis of suspension, the pendulum would be unlikely to remain in that position and would tip over and swing around, bringing the center of mass under the axis. The question arises as to why the boat does not do the same thing, since its center of mass is higher than the point of support. If the center of buoyancy were fixed in position with respect to the boat, the boat would certainly tip over, but as the boat rolls, the center of buoyancy swings from side to side. Fig. 78 shows the boat in another position. By observing the new shape of the displaced water, it becomes evident that the center of buoyancy is shifted to the right and the resultant vertical force due to the water now lies to the right of the center of mass. The force of gravity on the boat and the buoyant force now constitute a couple tending to produce counter-clockwise rotation, *i.e.*, tending to restore the boat to an upright position. If the boat were tipped the other way the center of buoyancy would shift to the left, and again the torque would tend to turn the boat to the upright position. The vertical line through the center of buoyancy when the boat is tipped to one side intersects the line ab at a point M, called the metacenter of the boat. If the center of mass C is below the metacenter, the boat is in stable equilibrium, and the stability is increased by increasing the distance between these two points. If the center of gravity were above the metacenter, the torque coming into action as the boat is tipped slightly to one side would be such as to tip it

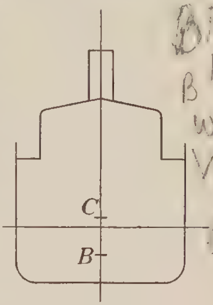


FIG. 77

BF = V
BF = w
w = V
Vρ =

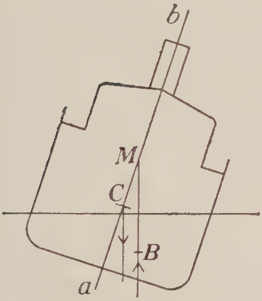


FIG. 78

farther from the upright position, and the boat would be capsized. In loading a ship the heavy part of the cargo is placed as low as possible, and when a trip is made without cargo, it is customary to carry ballast in the hold of the ship. Thus the center of mass is kept low and stability is assured.

120. Hydrometers.—The Principle of Archimedes, as applied to floating bodies, finds a further application in a piece of apparatus known as a hydrometer. It is used for measuring the density of liquids. It consists of a piece of glass tubing blown to a form similar to that shown in Fig. 79. The tube has a bulb at the lower end filled with shot or mercury which serves to hold it in an upright position when it is floating in a liquid. The upper part of the tube is graduated, so that the line of the scale that is at the surface of the liquid in which the hydrometer floats indicates directly the density of the liquid. To make the scale, the hydrometer is first put into pure water, and the point on the scale that lies at the surface of the water is marked one. Other points may be found by computation or by floating the hydrometer in other liquids of known density. Hydrometers of this kind are extensively used in determining the density of the solution in storage batteries. When a battery is fully charged the density of the solution is high, and when it is discharged the density is low. The reason for this will become apparent when storage batteries are studied.

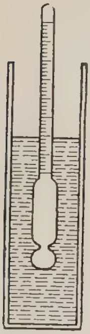


Fig. 79

121. Boyle's Law.—It has already been mentioned that if the pressure applied to a body of gas is increased, the change of the volume resulting will be very large as compared with the change of volume of a liquid resulting from an equal change of pressure. This statement is true unless the gas is in a condition approaching that in which it readily changes to a liquid. In 1662 Robert Boyle first announced the relation between the volume of a body of gas and the pressure to which it is subjected. This relationship is known as Boyle's Law, and is stated as follows: The volume of a body of gas varies inversely as the pressure to which it is subjected if the temperature remains constant. Suppose that a body of gas has a volume V_1 when its pressure is P_1, and another volume of V_2 when the pressure is changed to P_2; this law states that

$$V_1 : V_2 = P_2 : P_1,$$

from which it follows that:

$$P_1V_1 = P_2V_2.$$

This equation suggests that the law may be tested in the laboratory by measuring a series of volumes and the corresponding pressures of a body of gas, and seeing whether the product of pressure and volume remains constant as both are varied. Such an experiment must be conducted under conditions that assure a constant temperature. The relation of volume to pressure may be shown graphically by plotting the pressure as ordinate and the volume as abscissa, as shown in Fig. 80. The curve is a rectangular hyperbola.

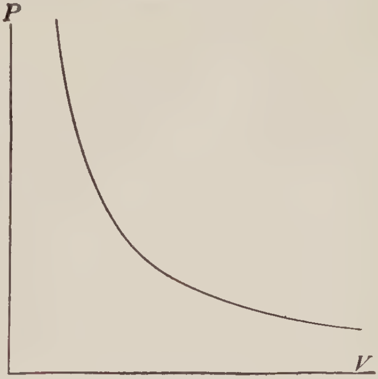


FIG. 80

If we plot the reciprocal of the volume against pressure, the graph should be a straight line, since the relationship is then one of direct proportionality. This is a convenient and satisfactory way of handling the data when the law is to be verified in the laboratory.

Density is defined as the mass per unit volume of the substance, *i.e.*, $d = M/V$, from which it follows that $V = M/d$. We may substitute this value of V in the equation $P_1V_1 = P_2V_2$, and the resulting equation is

$$\frac{MP_1}{d_1} = \frac{MP_2}{d_2}.$$

M may be canceled, and the proportion may be written in the form

$$\frac{d_1}{d_2} = \frac{P_1}{P_2},$$

which signifies that the density of a gas varies directly as the pressure to which it is subjected if the temperature is constant.

Any gas may be liquefied if its temperature is reduced sufficiently, and the amount of the reduction in temperature necessary to produce liquefaction is dependent upon the pressure applied to the gas, being greater as the pressure is less. As the conditions approach those under which liquefaction takes place, it is observed that the relation between the volume and the pressure of the gas departs more and more from that stated in Boyle's Law. For gases like hydrogen, oxygen, and nitrogen under ordinary conditions, the volume and pressure follow quite accurately the law of inverse proportionality; but gases like carbon dioxide and sulphur dioxide, which may be liquefied by applying high pressure at ordinary temperatures, show considerable departure from this relationship. A gas for which the relation of pressure to volume is *exactly* expressed by Boyle's Law is called a perfect gas. Such a gas is ideal and does not actually exist.

122. Atmospheric Pressure.—The atmosphere is a great body of fluid, and hence the propositions that relate to fluids in general apply in the case of the atmosphere. The pressure at a point in the atmosphere is the same in all directions. The atmospheric pressure against a surface is normal to the surface. The Principle of Archimedes applies to the atmosphere as to other fluids; hence a balloon in the air is buoyed up by a force equal to the weight of the air displaced. If the weight of the air displaced is greater than the weight of the balloon, the balloon will rise; and it will descend if its weight is greater than the weight of the air displaced.

Atmospheric pressure is due to the weight of the air, just as the pressure on a dam is due to the weight of the water. The magnitude of the pressure at any place expressed in pounds per square inch is the weight of a column of air one square inch in cross-section and having a length extending in the vertical direction as far as the atmosphere extends. When measured, this pressure proves to be about 14.7 pounds per square inch. This is called a pressure of one atmosphere and is sometimes used as a unit in terms of which pressure is measured; thus a pressure of 147 pounds per square inch would be called a pressure of ten atmospheres.

At high altitudes the weight of air above a given surface is less than at low levels, and the pressure is therefore less. The pressure and the density are directly proportional at a given temperature; hence, except in those regions where the temperature gradient is great, the density decreases as the altitude increases. When the pressure is one atmosphere and the temperature is at the freezing point of water, the density of air is .001293 gram per cubic cm. If the density of the air were uniform throughout and had the value .001293 gram per cubic cm, an atmosphere about five miles deep would produce the pressure that exists at the earth's surface; but, owing to the decrease of density with increase of altitude, the atmosphere must extend upward much farther than five miles. There is evidence of the existence of air more than eighty miles from the earth's surface.

A number of interesting experiments may be shown on the lecture-table illustrating atmospheric pressure.

123. The Barometer.—The barometer, as the name implies, is an instrument for measuring atmospheric pressure. The mercurial barometer is made as follows: A glass tube about a yard long is sealed at one end and left open at the other. The tube is filled with mercury and then heated till the mercury boils. The purpose of the boiling is to sweep out the remnant of air that clings to the walls of the tube. After the tube is freed from air, the open end is covered and the tube is inverted and placed with the open end in a bath of mercury. When the open end is uncovered under the mercury surface, the mercury in the tube falls till its upper surface is about 76 cm above the level of the mercury in the bath. A scale whose zero point may be adjusted to the level of the mercury in the bath is supplied for reading the height of the column in the tube. A common form of mercurial barometer is shown in Fig. 81. The height of the column is a measure of the atmospheric pressure. For accurate work it is necessary to make corrections for the effect of changes of temperature upon both the mercury and the scale. Just why the barometer furnishes a measure of atmospheric pressure is a question which should receive some consideration. If the tube were open at both ends, the mercury inside the tube would stand at practically the same level as that on the outside, since the pressure is the same at all points on the mercury surface. We may consider that a pressure P outside the tube is transmitted in accordance with Pascal's Law around into the tube, and that there is an upward pressure equal to P against the air in the tube. If the air were withdrawn from the tube by a pump, the mercury would be crowded up into the tube by the pressure outside, until the pressure hdg, due to the weight of the mercury, was sufficient to balance the pressure P due to the air. Since d and g may be regarded as constant, P varies as h, *i.e.*, the pressure is proportional to the height of the column. It is customary to consider the height, h, as a measure of the pressure and express gas pressures in terms of centimeters or millimeters of mercury. The height 76 cm is taken as a standard atmosphere. It is the average barometer reading at the sea level in latitude 45°.

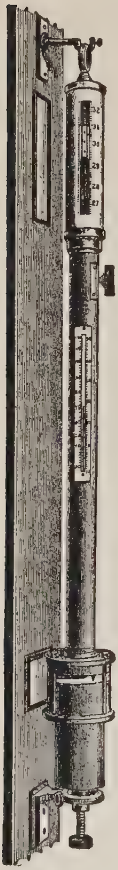


FIG. 81

To compute the pressure of one atmosphere in dynes per square cm, we find the product hdg, h being 76 cm and d being the density of mercury at 0° centigrade.

$$P = 76 \times 13.596 \times 980 = 1{,}012{,}630 \text{ dynes per cm.}^2$$

124. Coefficient of Elasticity of a Gas.—We have seen that the effect of change of pressure upon a gas is enormously greater than the effect of a corresponding change of pressure upon a solid or a liquid. It will be of interest to obtain more definite information regarding the magnitude of the coefficient of elasticity of a gas. Assume that the gas remains at constant temperature, and that its volume is V when it is under pressure P. Let p be a small change in the pressure, and v the resulting change in volume. Since the temperature is assumed to remain constant, the volume varies inversely as the pressure and

$$\frac{P}{P+p}=\frac{V-v}{V}.$$

Clearing this equation of fractions,

$$PV = PV - Pv + pV - pv,$$

from which

$$(P+p)v = pV,$$

and

$$\frac{p}{v/V} = P + p.$$

v/V is the relative change in volume and consequently is the strain corresponding to the change in stress equal to p.

Thus $\frac{p}{v/V}$, the coefficient of volume elasticity of the gas at constant temperature, is shown to be equal to $P + p$. For small changes of pressure p may be neglected; and the conclusion we reach is that the coefficient of elasticity at constant temperature is equal to the pressure. The coefficient of elasticity of air under ordinary conditions is about 14.7 pounds per square inch, while the coefficient of volume elasticity of steel is about 28,000,000 pounds per square inch.

The results of this topic will be found useful when computation of the velocity of sound in a gas is considered.

125. Pumps.—The action of the simple lift pump, Fig. 82, is dependent upon atmospheric pressure. As the piston is lifted, the pressure under it is reduced and the water is forced up the tube and into the pump by the atmospheric force exerted upon the surface of the water in the well. When the piston is pushed downward the valve *v* closes, and the water in the cylinder passes through valves *u* into the upper part of the cylinder. With the next upward stroke of the piston the water in the upper part of the cylinder is forced out of the spout, and the lower part of the cylinder is again filled from the well.

One atmosphere lifts mercury 76 cm, or about 30 inches; it should lift water 13.6 times as far, since the specific gravity of mercury is 13.6. This amounts to 408 inches, or 34 ft. In order to have water lifted to a height of 34 ft. by atmospheric pressure, it would be necessary that the pressure above the water in the pump be reduced to a value not greater than the pressure of mercury vapor above the mercury in the barometer. Experiment shows that water boils vigorously at ordinary temperatures when the pressure is reduced to the neighborhood of two cms of mercury and the space above the water is filled with vapor, which exerts a pressure upon the water. For this reason it is impossible to lift water 34 ft. with a lift pump, and the height to which it may be lifted is dependent upon the temperature of the water and upon the accuracy of construction of the cylinder, the piston, and the valves.

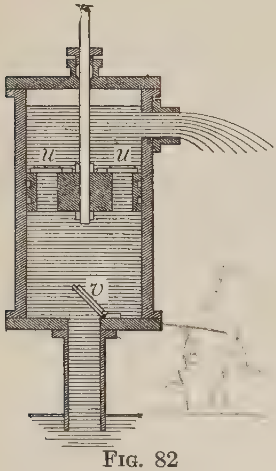


Fig. 82

When water is to be lifted much more than 25 ft. the force pump is used. It is placed near to the water surface, and the water is forced to the high altitude after it leaves the pump rather than before it enters the pump. Fig. 83 illustrates the construction and operation of such a pump. When the pump is started, some of the water enters the air chamber *A* and compresses the air that is confined there. On the upward stroke of the pump when no water is being delivered from the cylinder, the water in the air chamber is forced out by the compressed air; and thus the stream from the pump is steady instead of intermittent, as it would be if no air chamber were used.

In operating the pump of Fig. 83, most of the work is done on the downward stroke. This is undesirable, especially in the case of engine-driven pumps. In order to equalize the work done on the upward and downward strokes, the pump may be so built that it takes in water at both ends of the cylinder.

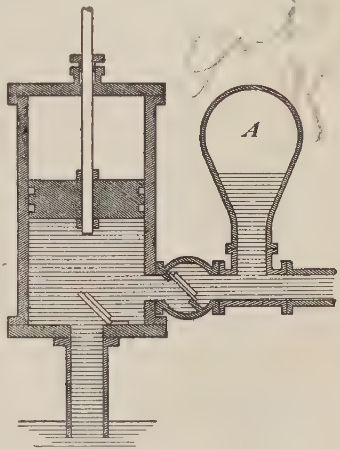


FIG. 83

PROBLEMS

NOTE.—A cubic foot of water weighs 62.4 lbs.

1. A tank is made of boards 10 ft. long and 1 ft. wide. One side of the tank consists of four such boards; hence the tank is 4 ft. deep. When the tank is full of water, find the force on each of the four boards composing one side.

2. A tank in the form of a cube 10 ft. on each edge is full of water. Imagine one side of the tank to be divided by a diagonal into two triangles. Find the force due to the water pressing on each of these triangles.

3. One of the two gates of a canal lock is 30 ft. wide. The water is 24 ft. deep on one side and 10 ft. deep on the other side. Find the force on the gate. Find the moment tending to turn the gate on its hinges, and the moment tending to turn it around a horizontal axis at the bottom of the gate.

4. A concrete dam is built with a triangular cross-section. The base is 6 ft. long, the height 8 ft., and the hypotenuse 10 ft. Make a sketch. (a) If the water is 8 ft. deep and presses against the vertical side, find the force per ft. of length of dam. 1996.8

5. If in problem 4 the water presses against the sloping side, find the force per unit length of the dam.

6. Find the moment per ft. of length tending to overturn the dam in problem 4 and also in problem 5.

per in

1995.

55

7. In problem 4 find the weight per ft. of length necessary to make the dam a stable structure.

8. A mill dam is 20 ft. wide and the water is 10 ft. deep. If the side upon which the water presses is vertical, find the moment of forces tending to overturn the dam.

9. A concrete dam is so built that its cross-section is a triangle having a horizontal base 8 ft. long. The side against which the water presses is inclined at an angle of 30° with the vertical, and measures 12 ft. from top to bottom. Sketch the cross-section. If the water comes to the top of the dam, prove that the moment of the resultant force tending to overturn the dam is zero.

10. A water-tight box is in the form of a cube 3 ft. on each edge. How much work would a man do in sinking it to the bottom of a pool of water 12 ft. deep? Weight of box is 100 lbs.

11. The specific gravity of brass is 8.4; what is its density in grams per c.c., and in pounds per cubic foot?

12. The cylinders of a hydraulic press are 12 inches and 1 inch in diameter, respectively. A pump handle 30 inches long is used to operate the piston in the small cylinder. The fulcrum is at one end of the lever, and the piston is fastened to the lever at a distance of 4 inches from the fulcrum. Treating the press as an ideal machine, find its mechanical advantage.

13. A cylindrical boiler 5 ft. in diameter contains steam at a pressure of 100 lbs. per square inch. What is the force per inch of length of the boiler tending to rupture the steel?

14. A glass stopper weighs 24 grams in air, 16 grams when immersed in water, and 9.6 grams when immersed in sulphuric acid. Find the density of the glass and of the sulphuric acid.

15. The government specifications for a life preserver require that it shall be able to exert continuously for 24 hours a buoyant force in excess of its own weight of 20 lbs. Cork has a specific gravity of .17. What volume of cork would be necessary to make a life preserver? Neglect the weight of the canvas.

07
005

16. Air at summer temperature weighs about .07 lb. per cu. ft., and hydrogen under the same conditions weighs about .005 lb. per cu. ft. A balloon weighing 250 lbs. and containing 9,000 cu. ft. of hydrogen could lift what additional weight?

17. Helium is twice as heavy as hydrogen, but it is not inflammable and hence is used in dirigible balloons. Making use of the figures of the preceding problem, determine how much larger a balloon must be if filled with helium instead of hydrogen in order that it may lift the same total load, including the weight of the balloon.

$W_A - ?$ $\frac{PV}{m} = $ [illegible]

18. A liter of air under standard conditions of pressure and temperature weighs 1.293 grams. If a liter of compressed air in a certain tank weighs 10 grams, what is the coefficient of elasticity of the air in this tank? The temperature is assumed to be 0° C.

$PV = P'V'$

19. A boat is floating in fresh water, and when a load of 500 tons is put on board it sinks 6 inches deeper. What is the area of the horizontal section of the boat at the water line? 32,051

20. An iron casting has a closed cavity within it. When it is put into water, the buoyant force due to the water is sufficient to support its weight. Compare the volume of the iron with the volume of the cavity. Density of iron is 480 lbs. per cubic foot.

1.6 [illegible]

200 lbs.

1.03

$BF = $ [illegible]

$= V \times 1.03 \times $ [illegible]

7.46 cu

stone & = 6, sp = 6 in air (300

[illegible] left under water

150 lbs in water 200

$6V = 7$ [illegible]

wt [illegible]

vol of material = [illegible] = 86.

130 − Y) [illegible]

Robt

CHAPTER IX

FLUIDS IN MOTION

126. Torricelli's Theorem.—This proposition deals with the velocity of the flow of a liquid through an orifice when the pressure causing the flow is due to the weight of the liquid. It is easily extended to include cases in which the pressure is produced in other ways. In Fig. 84 there is shown a tank having an orifice situated at a distance h below the surface of the liquid contained in the tank. The problem is to find the velocity V with which the liquid will flow from the orifice. The liquid in the tank has a certain amount of potential energy due to its position. If a small amount of liquid having a mass of m grams were to flow out of the orifice, the tank would appear just as it did before, except that the top layer containing m grams would have disappeared. Then the potential energy of the liquid would have decreased by an amount equal to mgh ergs. That amount of energy would be required to put the liquid back into the tank, since m grams of liquid would have to be lifted by a force of mg dynes through a distance h cm. The m grams of liquid flowing out of the orifice with a velocity V would have kinetic energy equal to $\frac{mV^2}{2}$. Assuming that the energy lost because of friction is insignificant, the potential energy lost from the tank would be equal to the kinetic energy gained by the outflowing liquid; then

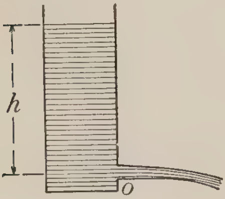


FIG. 84

$$mgh = \frac{m\,V^2}{2}.$$

Solving for V^2,

$$V^2 = 2gh \text{ and } V = \sqrt{2gh}.$$

P is p[illegible]
h in [illegible]
h in ft

This expression will be recognized as representing also the velocity gained by a body falling freely through a height h. Experiment shows that the velocity actually attained by a liquid flowing from an orifice does not differ from that given by the formula by more than 2%. The volume of liquid flowing from the orifice in a second is the product of the area of cross-section of the stream and the velocity, since the velocity represents the length of the column emerging from the orifice in a second. The volume flowing from the orifice in t seconds is AVt, if A represents the area of a right section of the stream.

The stream emerging from an orifice made in the thin wall of a tank contracts as it leaves the orifice, so that its area of cross-section at a short distance from the tank may be as much as 35% less than the area of the orifice. Fig. 85 suggests the reason for this contraction. The liquid, as it flows toward the orifice from different directions, acquires a momentum, one component of which is toward the center of the stream. The amount of the contraction depends upon the relation of the thickness of the wall of the tank to the diameter of the opening. If a tube having a length about three times its diameter is inserted in the opening, the emerging stream will contract only about 16%.

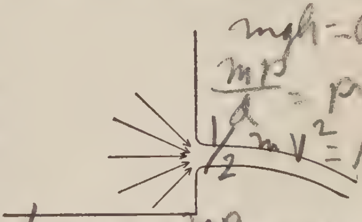


FIG. 85.

mgh = Potent[ial]

$\frac{mP}{d}$ = pressure

$\frac{1}{2}mV^2$ = K.E.

$\frac{1}{2}mV^2 + \frac{mP}{d} + mgh$ = Consta[nt]

127. Velocity of Efflux in Terms of Pressure.—The pressure at a depth h in Fig. 84 is represented by P and is equal to hdg. This value of P does not include the pressure due to the air. It is actually the difference in pressure between two points, one inside the tank at a depth h, and the other outside the tank. From the equation $P = hdg$, we find that $h = P/dg$. Substituting this value of h in the equation $V = \sqrt{2gh}$, it is found that

$P = hdg$

$$V = \sqrt{\frac{2P}{d}}.$$

$V^2 = \frac{2P}{d}$ where P is the pressure in [illegible]
on [illegible]

This equation is useful in determining the velocity of efflux when the pressure in the tank is known. The expression was derived from a consideration of liquids only, but there is no reason why it should not apply to fluids in general; hence the velocity of flow of a gas from a tank in which the pressure is in excess of the outside atmospheric pressure by an amount P is computed from the same formula.

128. Energy Relations in a Moving Fluid.—Fig. 86 represents a portion of a tube through which a fluid is moving from a toward b. a is a cross-section of the tube at right angles to the direction of flow. The mass per unit of volume, being the density, is designated as d, and the velocity of any small portion of the fluid is represented by v. Then the kinetic energy per unit volume is $\frac{1}{2}dv^2$.

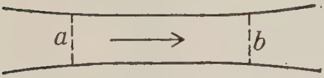


FIG. 86

If the position of the fluid is such that it may flow to lower levels and thereby do work, the system of which it is a part possesses dgh units of potential energy for each unit of volume of the fluid, h being the height of this portion of the fluid above some arbitrarily chosen level to which it is assumed the fluid might flow. Adding these two, we have the energy per unit volume of the fluid equals:

$$\frac{1}{2}dv^2 + dgh.$$

Assume p_a to be the pressure at a point in the surface a, and p_b the pressure at a point in the surface b. Consider the work done upon a unit of volume of the fluid in forcing it across the boundary a into the region ab. In §106 it is shown that the work done upon a fluid under conditions such as we are now considering is Pv. The work per unit volume is therefore P or, in the particular case under discussion, p_a. When this same unit of volume passes out of the region ab, it must do work upon the fluid beyond the boundary b equal to p_b units. The work done *upon* this unit of volume of fluid minus the work done *by* it equals the gain in energy, *i.e.*, $p_a - p_b$ equals the gain in energy of a cubic unit of the fluid in passing through the region *ab*. We may designate the energy per unit volume at *a* as $(\frac{1}{2}\,dv^2 + dgh)_a$ and that at *b* as $(\frac{1}{2}\,dv^2 + dgh)_b$; then

$$p_a - p_b = (\tfrac{1}{2}\,dv^2 + dgh)_b - (\tfrac{1}{2}\,dv^2 + dgh)_a.$$

129. Bernoulli's Theorem.—If a fluid is in motion through a pipe or in a channel, the motion is said to be steady if the velocity of flow at any point is the same at all times. In the case of steady flow, the path followed by a particle of the fluid is called a stream line. Each stream line remains fixed in position so long as the motion is steady.

The science of hydrodynamics is largely based upon the proposition known as Bernoulli's Theorem, which applies only to an incompressible and nonviscous fluid. The theorem is stated in the following equation:

$$\tfrac{1}{2}\,dv^2 + dgh + p = \text{a constant for points on a stream line.}$$

The truth of the proposition follows from the final equation in the preceding section. If we transpose the negative terms, that equation becomes

$$p_a + (\tfrac{1}{2}\,dv^2 + dgh)_a = p_b + (\tfrac{1}{2}\,dv^2 + dgh)_b.$$

The sum of the three terms is found to be the same for two points where a stream line intersects the boundaries *a* and *b*, but since we have not restricted the boundaries to any particular positions, it follows that this sum is the same for all points on a stream line.

130. Bernoulli's Theorem, Velocity Constant.—Bernoulli's Theorem is general in its application; but since there are three variables in the equation, it may be simpler to consider some special cases in which the number of variables is reduced to two. First consider the case in which the velocity is constant. Assume that a liquid is pumped through a long pipe of uniform cross-section. Since the fluid is incompressible, the velocity is uniform throughout the pipe. Suppose that the pipe were laid over a hill and through a valley, as illustrated in Fig. 87. Since *v*

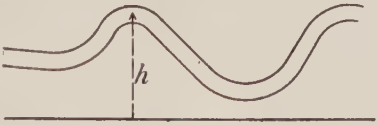


FIG. 87

is constant, Bernoulli's Theorem states that $dgh + p =$ a constant. The values of h are measured from some horizontal plane. According to the equation, p must decrease if h increases, and the pressure is a minimum at the highest point in the pipe and a maximum at the lowest point. We should hardly need a special theorem for reaching this conclusion since it follows from the results obtained in §108. In the succeeding paragraphs, however, we shall find that some new and interesting phenomena are most satisfactorily explained by making use of Bernoulli's Theorem.

131. Bernoulli's Theorem, *h* Constant.—If a pipe of variable cross-section is placed upon a horizontal plane, h may be considered as constant; and when a stream of liquid is forced through it,

$$\frac{dv^2}{2} + p = \text{a constant.}$$

Since the fluid is incompressible, its velocity must increase when it approaches the constricted part of the tube. If v increases, p must decrease if $\frac{dv^2}{2} + p$ is to remain constant. Thus we should expect to find that the pressure in the narrow part of the tube is considerably less than in the wider parts of the tube. Fig. 88 represents a piece of apparatus which furnishes an experimental demonstration of this conclusion. As water is forced through the horizontal tube it rises in the two small vertical tubes at either end, showing that the pressure at these places is greater than atmospheric pressure. At the constricted part of the tube no such action is observed. The tube from here is bent around and its lower end inserted in a dish of water; the water from this dish rises in the tube, showing that the pressure in the constricted part of the tube is less than atmospheric pressure.

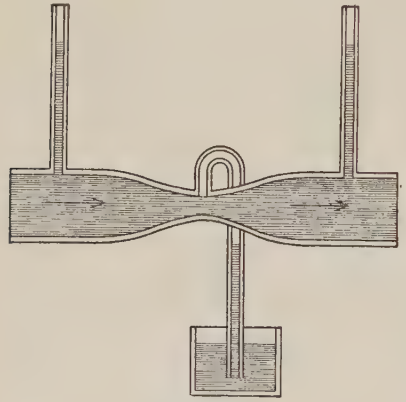


FIG. 88

132. The Water Aspirator.—In the water aspirator or jet pump shown in Fig. 89, the principle discussed in the preceding paragraph finds a practical application. The aspirator is connected at its upper end to a faucet furnishing water from the city pipes. The tube is greatly constricted at the point *B*, and the pressure is reduced far below atmospheric pressure. The vessel to be evacuated is connected to the aspirator by the tube *C*. Surprisingly low pressures may be produced by well designed pumps of this kind. Some of them will pump mercury to a height of 70 cm.

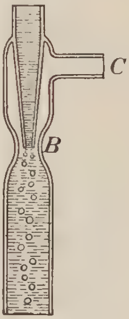


FIG. 89

133. Gases in Motion.—The equation representing Bernoulli's Theorem does not apply to gases because the volume of a gas changes when the pressure changes; nevertheless, qualitatively, the effects that we have been studying in the last two paragraphs may be produced if we substitute a gas for the flowing liquid. The ordinary atomizer furnishes a familiar illustration.

If a jet from an air compressor is blown vertically upward, a rubber ball placed in the path of the jet will be supported by the jet in stable equilibrium. As the ball moves slightly to one side of the jet, most of the air passes the ball on the side next to the middle of the jet. The path of the jet is obstructed by the ball, and thus the air must flow through a constricted area as it flows around the ball. This results in a reduced pressure on the side of the ball next to the jet, and the ball is pushed toward the middle of the jet by the greater pressure on the other side.

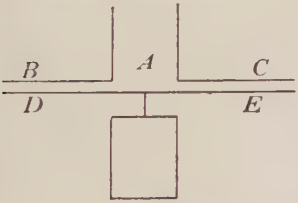


FIG. 90

In Fig. 90, *A* represents a metal tube fitted into a hole in a plate represented as *BC*. A second plate *DE* is placed under the first one and quite near to it. If a blast of air is blown downward through the tube *A*, upon reaching the plate *DE* it is deflected outward and must flow through a passage of reduced area. Thus the pressure between the plates is lowered and may become less than atmospheric pressure. The air pressure on the under side of plate DE may exceed the pressure on its upper surface by an amount sufficient to support the weight of the plate and an additional load of two pounds or more.

The steam injector is a piece of apparatus for pumping water into a boiler; its construction is similar to that of the jet pump shown in Fig. 89. Steam under pressure from the boiler is blown through the vertical tube, and the tube C is connected to a pipe running down into a well. The pressure in the pipe is so reduced that the water in the well is forced up through the pipe by atmospheric pressure and flows into the injector. Here it is caught in the jet of steam and carried out through the pipe at the bottom. The operation is not the same as when air is blown through the tube, since the steam coming in contact with the cold water is suddenly condensed; thus there is an additional cause of the reduction of pressure. The steam comes through the nozzle with very high velocity and when it is condensed in the water, sufficient momentum is given to the water to enable it to force itself against the steam pressure into the boiler from which the steam came. This result is possible because of the condensation of the steam; the injector will not pump hot water into the boiler.

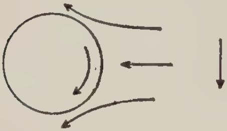


FIG. 91

The curving of a baseball is explained as a result of a pressure difference due to the difference in velocity of the air on the two sides of the ball. Let the circle in Fig. 91 represent a ball having a velocity to the right and at the same time a rotation as indicated by the arrow within the circle. So far as pressure is concerned, the effect is the same whether the ball moves through the air or whether it is stationary and the air flows past it. The arrows, one above the path of the ball and the other below it, indicate the motion of the air relative to the ball. On the upper side the motion of the air is impeded by friction against the ball. On the lower side the rotary motion of the ball is in the same direction as the relative motion of the air, and the air is either accelerated or only slightly retarded by its contact with the ball. Thus the velocity of the air with respect to the center of the ball is greater on the lower side than on the upper side, and therefore the pressure is less below the ball than above it. The ball then experiences a downward force due to the air pressure, and it will follow a path having a downward curvature greater than that produced by gravity alone. When the axis around which the ball spins is vertical, the ball curves to the right or to the left, depending upon the direction of rotation.

PROBLEMS

1. A steel tank contains water to a depth of 16 ft. A hole is drilled through the side of the tank at the bottom, and a short cylindrical tube fitted to it. The hole in the tube has a cross-sectional area of 1 sq. in. Find the velocity of efflux of the water, and the number of cubic feet per minute discharged. See §126.

2. Find the pressure necessary to throw water from a fire hose to a height of 96 ft. With what speed does the water leave the hose if it rises vertically? Neglect friction.

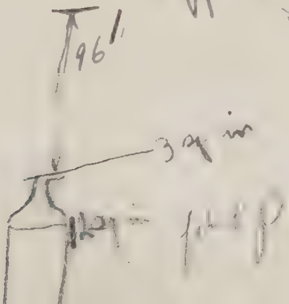
3. Water coming from an orifice 25 ft. from the ground starts in a horizontal direction and has a range of 36 ft. Compute the pressure in the tank near the orifice. Neglect friction.

4. Water is pumped through a horizontal pipe having an area of cross-section of 5 cm² with a speed of 200 cm per second. The pressure in this part of the pipe is 2 x 10⁶ dynes per cm². If the pipe has a variable diameter, what would be the pressure at a place where the area of cross-section is 1 cm²? $V_1 A_1 = V_2 A_2$ $V = 1000 \frac{cm}{sec}$

5. In the pipe system of problem 4, a faucet is opened below the parts under consideration, causing the velocity of flow to increase by 50%, while the pressure in the wide part of the pipe remains unchanged. Compute the pressure at the constricted section. What change in the area of cross-section of the constricted part would compensate for this increase of velocity and leave the pressure the same as in problem 4?

$p_1 = 420,000$ [illegible]

$\frac{V_1}{1000} = \frac{1}{a} = 1.4$



CHAPTER X

MOLECULAR PHENOMENA

134. Surface Tension.—We come now to the discussion of some phenomena which apparently contradict the laws of hydrostatics we have been studying. A detailed description of the phenomena suggested by Fig. 92 will serve to illustrate what is meant.

A shows a drop of mercury resting upon a smooth board. Observations of liquids under ordinary conditions lead to the conclusion that the free surface, when the liquid is at rest, is a horizontal plane; yet here is a case in which the surface is almost spherical if the drop is small.

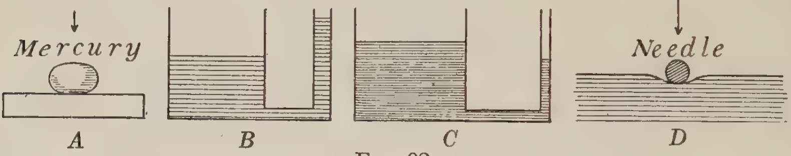


FIG. 92

B represents a vessel having a horizontal base and consisting of two communicating tubes. If one of them is sufficiently small and water is put into the vessel, it immediately rises to a higher level in the small tube than in the large tube. The pressure on the bottom of the vessel is everywhere the same and is expressed by the equation $P = hdg$. In this case there are two values of h, the one obtained from a measurement in the small tube being considerably larger than the one obtained from the large tubes. Which h should be used in computing the pressure, and why do they differ?

In *C* there is shown a similar vessel containing mercury. The mercury level in the small tube is depressed.

In *D* a steel needle is pictured as floating on the surface of water. Steel is seven times as dense as water and, by the

Principle of Archimedes, it would seem that the needle should sink in the water.

These and other apparently anomalous phenomena are easily explained when we come to understand that there is a tension in the boundary or free surface layer of a liquid, which is one of the manifestations of molecular attractions.

135. Molecular Forces.—The results of molecular forces are everywhere apparent. Without them, matter could not exist either in the solid or in the liquid states. The names cohesion and adhesion are used to designate molecular forces, cohesion signifying the force between like molecules and adhesion the force between unlike molecules. The fact that molecular forces have an appreciable magnitude only when the molecules are extremely close together is familiar. A rod of steel one square inch in cross-section may require a force of 60,000 pounds in the direction of its length to break it; but when the two parts are put together so accurately that it is difficult to see where the break occurred, no forces tending to hold the parts together are detected. The molecular range is the name applied to the maximum distance between molecules that show any molecular attraction. There is evidence to show that this range is considerably less than a millionth of a cm.

It is difficult to form any judgment as to the comparative magnitudes of molecular forces in liquids and solids. A body of liquid is easily separated into two parts, but this is because the liquid flows under the action of the forces applied; it does not signify that it would be easy to tear the molecules of one part from those of another without first reducing the size of the section where the break is to occur. Certain experiments show that the molecular forces in liquids are as large as those in many solids.

136. Explanation of Surface Tension.—In order to understand why the surface of a liquid contracts to the smallest area possible under the given conditions, let us consider a mechanical model which we may endow with some of the properties of a liquid. Imagine a large tank filled with spheres about the size of baseballs, and that there is little or no friction at the surfaces where they touch one another. We assume an attraction between balls that lie near together, perhaps within a distance equal to two or three diameters. There is also assumed to be an attraction between the walls of the tank and those spheres that lie near it. Suppose further that the walls of the tank are somewhat flexible. If at a given instant the surface layer were made up of spheres whose centers all lie in the same horizontal plane, there would be no tendency for one to behave differently from the others; but if one, such as c in Fig. 93, should in some way get out of the horizontal plane, a and b and others similarly situated would be drawn together by their mutual attraction and would close in over c, crowding it out of the surface. Thus the area of the surface is slightly reduced and the walls of the tank, which are assumed to be flexible, are bent inward slightly. There is practically no opposition to the operation here described, since the attraction between neighboring spheres is enormously greater than the weight of one of the spheres and the hydrostatic force tending to push c upward is insignificant. To carry the analogy farther, we may assume that the spheres are in motion, as we have positive evidence that the molecules of a liquid are always in motion. Then some of the spheres will be continually getting out of the horizontal plane, making it possible for others to close in over them and crowd them out of the surface layer. Thus there are always present the conditions which will result in a diminution of the area of the free surface if such diminution is possible.

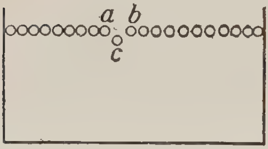


FIG. 93

Of course, this is a crude model; molecules are probably not spherical, they are not in contact in the way that the spheres were imagined to be in contact, their motions are rapid and irregular, and they are in contact only during collisions. Even the collisions may not be simple mechanical operations like the collision of two elastic balls. It is very unlikely that the surface layer of molecules would have the perfectly regular arrangement pictured in Fig. 93. Yet, with all these differences, it is believed that the contraction of the surface layer of a liquid is essentially like the process described as operative in this model.

The surface of the drop of mercury in Fig. 92 tends to contract, and, since a spherical body has less superficial area for a given volume than a body of any other form, the mercury tends to assume a spherical shape. It is, however, somewhat flattened by the action of gravitational forces and the reaction of the board upon which it rests. Dewdrops are very nearly spherical. Since they are small, the forces acting upon them are small and they are not appreciably distorted.

In the manufacture of shot, molten lead is poured through a fine sieve at the top of a tower, and thus it is broken up into small particles which take on a spherical form because of surface tension. These globules cool and solidify during their fall, and then drop into a tank of water at the bottom of the tower. Thus the spherical shape is preserved.

The needle floats on the surface of the water because it makes a depression in the water, and the forces due to surface tension have a vertical component sufficiently large to support the weight of the needle.

137. Measurement of Surface Tension.—Since the free surface of a liquid tends to contract, it is evidently under tension at every point. At a given point the tension is tangential to the surface and is the same in every direction in the tangent plane. The force per unit width of the surface measured at right angles to the force is the measure of the surface tension, and is designated as T. To measure the surface tension of a liquid, a Jolly balance and a little frame of platinum wire may be used. The frame is first heated in a flame to burn off all impurities, and then it is suspended from the

Why tendency of the surface [illegible]

spring of the balance and dipped into the liquid. Fig. 94 illustrates in outline the form of the apparatus. When the spring is lifted a little, the frame is pulled partially out of the liquid and a film is formed which exerts a downward force on the frame. This force is measured by the Jolly balance, and is represented by F. There is a surface tension in all directions in the plane of the film, but it is only the vertical force that stretches the spring. The horizontal force merely tends to draw the opposite sides of the frame together, but the frame is sufficiently rigid so that the force produces little effect. If L represents the width of the frame, *i.e.*, its horizontal dimension, $2L$ is the total width of surface, since the liquid film has two surfaces. The force per unit width of the surface is then

$$T = \frac{F}{2L}.$$

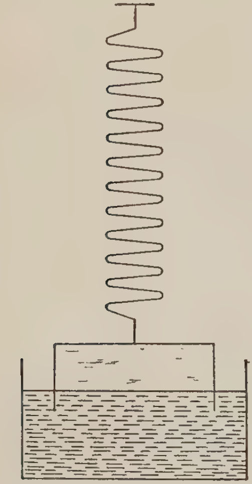


FIG. 94

For pure water the surface tension varies from 70 to 80 dynes per cm, depending upon the temperature of the water. Students making these measurements for the first time are usually surprised at finding the surface tension of soap films only about one-third as great as that for pure water. The surface tension of mercury is more than 500 dynes per cm.

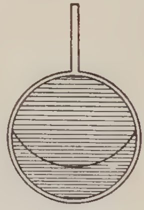


FIG. 95

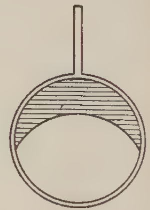


FIG. 96

138. Further Illustrations of Surface Tension.—Suppose that a thread were tied across a circular wire frame, as shown in Fig. 95, and the frame then dipped into soap solution. If the frame were then removed from the liquid, a film of the solution would remain on the frame and the thread would hang as in the figure. If the film below the thread were broken, the surface tension in the part of the film remaining above the thread would lift the thread into the position shown in Fig. 96.

Fig. 97 illustrates a similar frame with the thread in the form of a loop. When the film inside the loop is broken, the thread assumes a circular form. The largest possible area inside the loop is attained when the shape is circular; and in this condition the area of the film outside the loop is a minimum.

That different liquids have very different surface tensions is illustrated by the following experiments: Float two thin splinters of wood side by side on the surface of water. Place a drop of alcohol on the surface between them. The splinters are quickly drawn apart because the surface tension of alcohol is less than that of water and they move in the direction of the greater forces.

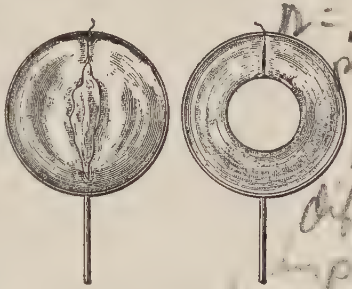


Fig. 97

Another experiment illustrating the same thing consists in putting a small piece of gum camphor on the surface of clean water. The camphor dissolves slightly in the water, and because of the irregularity in shape of the piece, it dissolves unequally on different sides. This results in inequalities of surface tension, and the particle is set in rapid motion.

When a drop of oil or gasoline is put upon the surface of water, forces resulting from surface tension are brought into action, which may be represented as in Fig. 98. Let T_w be the surface tension of water, and T_o the surface tension of the oil. The tension in the surface under the oil drop may be represented by T_{ow}. It so happens that T_w is greater than the sum of T_o and T_{ow}; hence there can be no equilibrium of the surface forces acting at the edge of the drop. The result is that the drop is drawn out in all directions until it covers the entire surface of the

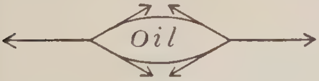


Fig. 98

water, or, if the body of water is large, until the layer of oil is so thin that it can no longer be detected. When the thickness of the oil film becomes comparable to the wave length of light, colors due to the interference of light may be observed.

It is proverbial that water cannot be carried in a sieve; but this is not always true. If the sieve has a fine mesh and is made in the form of a cylinder closed at both ends, it may be filled with water and carried around the room without a leak. If the surface film is broken by blowing upon the upper side of the cylinder, all the water immediately runs out.

139. Angle of Contact.—If a clean glass plate is held with its surface in a vertical plane and then dipped into water, it is found that the surface of the water does not meet the glass surface at right angles but that it curves upward, so that the two surfaces are practically parallel at the line of contact. This is described by saying that the angle of contact is zero. When there are impurities in the water, the angle at which the two surfaces meet is no longer zero; it may be 20° or more. If a smooth block of paraffin is substituted for the glass, the surface of the water will curve downward instead of upward. The angle of contact in this case is greater than 90°. Similarly, a mercury surface has a downward curvature where it meets a vertical glass surface. Fig. 99 illustrates the two cases just described. α is the angle of contact in each case. Whether the surface of the liquid curves upward or downward depends upon the relative magnitudes of the molecular forces involved. Water is attracted by glass so strongly that if a piece of glass is put into water and then withdrawn, its entire surface is found to be coated with a thin layer of water. This is only true if the glass is clean. Under these conditions the water is said to wet the glass. Tracing the water surface from right to left in the figure, when we come to the glass the surface bends upward since the surface of the glass is coated with water.

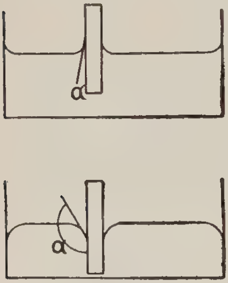


Fig. 99

Since the surface tends to contract, it takes a form like that shown in the figure.

With mercury the case is different. The surface tension of mercury is seven times as great as that of water, and the attraction of glass for mercury is less than the attraction of mercury for mercury; hence the mercury does not cling to the glass. In other words, the mercury does not wet the glass. Tracing the mercury surface in the lower part of Fig. 99, when we come to the glass plate there is no mercury on the plate; hence the surface turns downward along the surface of the glass. A contraction of the mercury surface tends to depress the mercury near the plate.

140. Liquids in Capillary Tubes.—As explained in the preceding topic, water in contact with glass rises slightly higher at the line of contact than the general level of the water elsewhere in the vessel. Suppose that two parallel plates are dipped into water, as shown in Fig. 100. If the distance between the plates is reduced, the cross-section of the supported column is reduced and the forces due to surface tension can support a correspondingly higher column; thus the height of the column is inversely proportional to the distance between the plates.

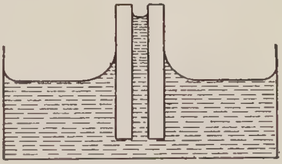


FIG. 100

Returning to the question raised in §134 regarding *B* and *C* of Fig. 92, we now have the explanation of the higher level of the water in the small tube. The water wets the walls of the tube, and the surface, in contracting, is lifted. This is analogous to lifting the piston in a pump. The pressure beneath the surface is decreased and the atmospheric pressure forces the water up, just as the water is forced upward under the rising piston of the pump. However, if there were no atmospheric pressure the column would still be supported at the same level, but the water in the tube would be under tension and not under pressure. The weight of the water would then be supported by a force pulling from above, rather than by a force pushing from beneath. This is possible in a capillary tube but not in a pipe such as would be used in connection with a pump. A pump would not operate in a vacuum. That water in a capillary tube is capable of transmitting a tension is proved by experiment.

The pressure of the air above the surface is partially supported by the forces in the surface layer, and hence the hydrostatic pressure immediately under the surface is less than atmospheric. Pressure equal to atmospheric pressure occurs in the small tube at the level corresponding to the level of the liquid surface in the large tube. The expression $P = hdg$ correctly represents the pressure on the bottom of the vessel, provided that h represents the depth of the water in a part of the vessel where surface tension does not alter the depth, and provided that P represents the pressure in excess of atmospheric pressure.

The depression of the mercury in the capillary tube now needs no further comment.

141. Height of the Column in a Capillary Tube.—To compute the height of the column when the liquid rises in the capillary tube above the normal level, we equate the weight of the column to the resultant vertical component of the forces due to surface tension and solve the equation for h, the height of the column. Let T be the surface tension, r the radius of the tube, d the density of the liquid, α the angle of contact, and g the acceleration due to gravity. The surface film lines the interior of the tube above the column, and the line across which the forces act is the circumference of the tube. T is the force per unit width of the surface measured along the circumference, and $T \cos \alpha$ is its vertical component. $T(\cos \alpha)\, 2\pi r$ is the resultant upward force in dynes. The volume of the liquid supported by this force is $\pi r^2 h$, and its weight is $\pi r^2 hdg$ dynes. Equating these two forces,

$$\pi r^2\, hdg = T(\cos \alpha) 2\pi r.$$

Solving for h,

$$h = \frac{2\, T \cos \alpha}{rdg}.$$

For pure water against clean glass, $\alpha = 0$ and $\cos\alpha = 1$. Taking $T = 75$ dynes per cm, we find for a tube .2 millimeter in diameter

$$h = \frac{150}{.01 \times 980} = 15 \text{ cm approximately.}$$

The equation just derived expresses the fact that the height of the column in a capillary tube varies inversely as the diameter of the tube, and is directly proportional to the surface tension of the liquid. If the surface tension is first measured by the method described in §137, the equation suggests a way of determining the angle of contact, α. The diameter of the tube is measured with a micrometer microscope, and the height of the column is so large that its measurement offers no difficulty. Sometimes the equation is used as a means of computing the radius of a capillary tube. The height h of a water column in the tube is measured. The surface tension of water is known, and r is the only unknown in the equation.

142. Capillary Action and Floating Bodies.—It is frequently observed that if two small bodies are floating in a glass of water, they show a tendency to come together and remain together. This is explained as a result of capillary action. If the water wets the bodies, it rises slightly on all sides where it makes contact with the bodies and it rises higher in the space between the bodies than elsewhere, as illustrated in Fig. 100. It was mentioned in §140 that the pressure between the plates of Fig. 100 is less than atmospheric pressure at points in the liquid above the general level of the water in the vessel. On the outside surfaces of the plates, however, the pressure above the water line is the atmospheric pressure. Thus it is seen that the forces pushing the plates together is greater than the forces tending to separate them.

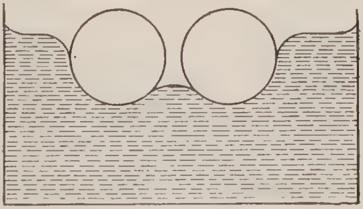


Fig. 101

If the liquid does not wet the objects and is depressed, as in Fig. 101, the bodies are also forced together, since between them the pressure above the liquid surface is due to the atmosphere only; whereas, on the outside, there is a hydrostatic pressure in addition to the atmospheric pressure over a larger portion of the surface.

Two floating bodies will be *separated* one from the other if one of them is wet by the liquid and the other is not. The reader can easily convince himself of the truth of this statement if he will draw a figure representing the conditions.

143. Diffusion of Gases.—The student of physics and chemistry is familiar with the molecular theory of the construction of matter, but he is likely to view it with skepticism unless he can find some positive evidence of its truth. It is quite impossible, in a beginning course in physics, to deal quantitatively with the different aspects of the molecular theory, nor is it possible to discuss the problems related to atomic structure; but if only a few items of evidence can be presented here, they may furnish an incentive to look for further proofs in the fields of advanced physics.

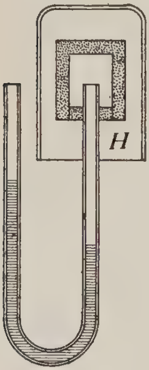


FIG. 102

The following experiment furnishes evidence of molecular motion which no one will fail to appreciate. A porous cup made of unglazed earthenware is fitted with a stopper through which a glass tube passes. The tube is U-shaped and is partly filled with colored water, as shown in Fig. 102. The air pressure within the cup is the same as that outside, so that the water in the two branches of the U-tube stands at the same level. If an inverted beaker containing hydrogen or illuminating gas is placed over the porous cup, the water levels in the U-tube immediately begin to change, showing an increase of pressure in the cup. After a few seconds, if the beaker is removed, the water in the U-tube moves in the opposite direction and soon indicates that the pressure within the cup is less than the atmospheric pressure outside. The conclusions to be drawn from this simple experiment are important. The motion of the gas is not due to a difference in total pressure, because it went through *against* a pressure that was greater on the inside than on the outside of the cup. The kinetic theory of gases furnishes an adequate explanation of the experiment. A gas consists of very small particles or molecules; each molecule is in rapid motion, its motion being in a straight line until it collides with another molecule or with some other body. The collisions occur very frequently so that the motion of the molecule is irregular. As a result of this rapid and irregular motion, some of the molecules find their way through the pores of the porous cup. The hydrogen molecules, being much lighter than the molecules making up the air, move with greater speed; and hence the number of them finding their way through the walls of the cup is greater than the number of molecules of the heavier gas passing through in the opposite direction. This accounts for the increase of pressure within the cup.

After considerable hydrogen has accumulated in the cup, the beaker is removed. We then have hydrogen inside the cup and air on the outside, so that the conditions are reversed. The hydrogen diffuses outward faster than the air diffuses inward, and very soon the pressure in the cup is less than atmospheric pressure, as shown by the difference in levels of the water surfaces in the U-tube.

It may be shown by experiment, and also from theoretical considerations, that the rates of diffusion of two gases are to each other inversely as the square roots of their densities. Thus, hydrogen diffuses four times as fast as oxygen, since their densities are in the ratio of 1 : 16 if their pressures are equal.

144. Diffusion of Liquids.—The evidence of molecular motion in liquids is not so striking as in the case of gases, but it is nevertheless convincing. If a dense solution of copper sulphate is put at the bottom of a deep jar and pure water is put over it by some process that avoids mixing the two, the copper sulphate will diffuse upward and the water will diffuse downward so that after a long time the solution will seem to have a uniform density throughout. This operation goes on in opposition to the action of gravity, which would tend to maintain the initial conditions. It may take several weeks to reach a state approximating uniform density. The slowness of the operation seems to indicate less freedom of molecular motion in a liquid than in a gas.

Even in the case of solids there are satisfactory reasons for believing that each molecule is in rapid and irregular motion, but its motion is restricted to a very limited region; it is not free to wander through the entire space occupied by the body as in the case of liquids and gases.

145. The Brownian Movement.—In 1827 Brown, an English botanist, observed that when very fine particles are held in suspension in water, the particles are continually in motion. The motion of a single particle is utterly irregular; it darts first in one direction, then in another, sometimes with one speed, sometimes with another. The activity is greater if the particles are extremely small; hence the phenomenon is observed to the best advantage only under a high-power microscope. An alcoholic solution of gum mastic in water has been used extensively for studies of the Brownian movement; but the water-color paints, gamboge and yellow ochre, are more satisfactory when one merely wishes to observe the phenomenon. The paint is mixed with water and put upon a microscope slide, then covered with a very thin cover glass and viewed through a microscope with an oil immersion lens. The explanation of the phenomenon is as follows: a molecule of water is in ceaseless motion, it collides with another molecule, rebounds, and collides with another and so on indefinitely. Its motion is utterly irregular, both as to direction and speed. If a particle of matter that is large enough to be seen with the microscope is floating in the water, it will be bombarded by the moving molecules and will have a motion similar to that of the molecules. Its mass is enormously greater than that of a molecule so that its velocity will be small. The motion is due to the fact that with a perfectly random distribution of the impacts of the molecules upon the particle, the many impulses applied to it in a very brief interval of time are not likely to be balanced. In another brief interval the resultant impulse will have another direction and so the motion is erratic like that of the molecule itself. If the water did not have a molecular structure and were a continuous medium, or if its particles were infinitely small, no such effect as the Brownian movement would occur. For this reason the discovery of this phenomenon was of profound significance, proving the existence of the granular structure of matter and, many years later, leading to a method of determining the number of molecules in a cc of gas.

146. Avogadro's Constant.—The student of chemistry is familiar with the proposition known as Avogadro's Law, which states that under the same conditions of temperature and pressure, equal volumes of different gases contain equal numbers of molecules. This proposition is fundamental in the chemistry of gases, and is a vital part of the kinetic theory.

Since the number of molecules per cubic cm is the same for all gases, this number becomes an important physical constant. It is expressed as the number of molecules per cubic cm at the temperature 0° centigrade and a pressure of 760 mm of mercury. The French physicist Perrin, by an elaborate research based upon observations of the Brownian movement, reached the conclusion that this constant is 3.05×10^{19} molecules per cubic cm. Other totally different methods have given the value as 2.70×10^{19} molecules per cubic cm. The latter value is considered the more accurate. The mind fails to grasp the enormous magnitude of this number. It may be somewhat more expressive to state it as 27 million million million million molecules per cubic cm. Since one gram molecule of any gas occupies 22.4 liters, the number of molecules per gram molecule is $2.7 \times 10^{19} \times 22{,}400 = 6.06 \times 10^{23}$. This is the number that is usually called Avogadro's number.

147. Solutions.—It is difficult to formulate a concise definition of a solution that is free from objection. Ostwald defined solutions as "homogeneous mixtures which cannot be separated into their constituent parts by mechanical means." The term "homogeneous" is intended to convey the idea of an intermingling of particles of molecular dimensions rather than of particles of larger size, but it is objectionable because it seems to exclude cases in which there are different concentrations in different parts of the solution. The term solution is a broad one. Solids, liquids, or gases may dissolve in a liquid to form a liquid solution. If two gases are mixed, one may be considered as dissolved in the other. Sometimes the term solid solution is used. Iron carbide is said to be dissolved in iron to form steel. When a solid substance is dissolved in a liquid, the liquid is called the solvent and the dissolved substance the solute. In the discussion immediately following, we shall deal only with liquid solutions in which no chemical action takes place as a result of the contact of solvent and solute. Cases in which the solute is ionized when put into solution are excluded.

When a crystalline substance, such as sugar, is dissolved in water, the crystalline structure is broken up and the molecules are separated one from another. In this condition they have a freedom of motion similar to the freedom of gas molecules. We conceive that the pressure of a gas upon the walls of the containing vessel is due to the bombardment of the molecules upon the walls; and since the sugar molecules in solution have both freedom and motion, it ought to be possible to show that a pressure exists in the solution due to the presence of the sugar. Furthermore, this pressure should prove to be proportional to the concentration of the solution, *i.e.*, to the number of molecules of sugar, per unit volume of solution. This statement is analogous to Boyle's law for gases, since Boyle's Law may be stated as follows: At constant temperature the pressure of a gas is directly proportional to its density.

The boundaries of the liquid limit the volume occupied by the sugar, just as the walls of the containing vessels limit the volume occupied by the gas. Part of the boundary of the liquid is usually its free surface upon which there is atmospheric pressure only; hence it is apparent that if there is a pressure within the solution that is proportional to the concentration, it is not exerted against the walls of the containing vessel but rather against the cohesion of the solvent. A gas confined in a closed vessel has a tendency to expand indefinitely, and is prevented from doing so by the forces exerted by the walls of the containing vessel. The dissolved substance likewise tends to expand indefinitely, but is confined to the volume occupied by the solvent and is prevented from expanding by the forces that hold the parts of the solvent together. The pressure that we have here assumed to exist is called osmotic pressure. We shall return to a further discussion of the subject in another paragraph.

148. Osmosis.—We have seen that gases diffuse rapidly through porous partitions, and we have found also that liquids, while they diffuse more slowly, show clearly that their molecules are in motion. Two liquids separated by a porous partition will diffuse through it so that after a time a mixture of the two liquids is found on both sides of the partition. Usually the rates of diffusion of the two liquids are unequal, and in some cases the porous partition permits one liquid to pass through it but not the other. Alcohol will pass through rubber but water will not. However, this is probably not a true case of diffusion. If a solution is put into a porous cup, it may happen that the solvent will diffuse through the walls of the cup and the solute will not. Such a porous partition is called a semi-permeable membrane for the particular substance involved. The best semi-permeable membranes have been made by filling the pores of unglazed porcelain cups with a precipitate of copper ferro-cyanide. A piece of parchment serves fairly well as a semi-permeable membrane for a water solution of sugar; water diffuses freely through it while the sugar is not permitted to pass. If a sugar solution is put on one side of the parchment and pure water on the other side, water will diffuse through the membrane toward the sugar solution faster than the water in the solution will diffuse in the opposite direction; thus there is a tendency to dilute the solution. This general process involving unequal rates of diffusion through a porous membrane is called *osmosis*, and operations depending upon this process are said to be due to osmotic action. A cell illustrating osmotic action may be made by fastening a sheet of parchment over the end of a thistle tube, as shown in Fig. 103. The tube is partially filled with a sugar solution and suspended in a beaker of pure water. The unequal rates of diffusion of the water in the two directions through the membrane are shown by the gradual rise of the surface of the solution in the tube. It should be kept in mind that, so far as the membrane is concerned, the water may pass through it with equal ease in either direction. It is the presence of the sugar that gives rise to the inequalities in the rates of diffusion.

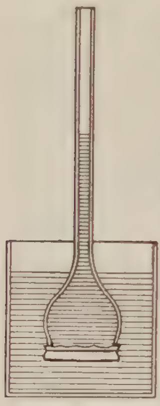


FIG. 103

149. Explanation of Osmotic Action.—In order to explain the phenomenon, let us imagine, instead of the thistle tube, a cylinder open at both ends. Over the lower end there is fastened a piece of parchment or other semi-permeable membrane, and a piston is fitted to slide inside the cylinder as indicated in Fig. 104. The cylinder is supported with its lower end dipping beneath the surface of water, and a load L is placed on the piston. If the cylinder itself contains pure water, the piston will gradually fall, since the membrane does not keep the water from diffusing out of the cylinder under the action of the pressure due to the load. The pores of the membrane are filled with water so that there is no discontinuity between the water above the membrane and the water below it, *i.e.*, there is no boundary or water surface at the membrane. If the contents of the cylinder were a sugar solution, and if the membrane is impervious to sugar, the concentration of the solution would increase as the piston descends and the pressure would ulti-

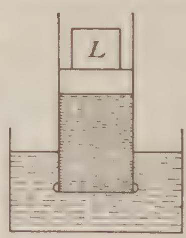


FIG. 104

mately become sufficient to support the load so that it would be brought to rest. The water does not assist in supporting the piston because, so to speak, it has nothing to stand on. If now we suppose that the load is removed from the piston, the force tending to raise the solution and the piston is greater than their combined weight and the piston will be lifted as rapidly as the water can flow through the pores to fill the increasing volume. This process will go on till there is developed at the membrane a hydrostatic pressure, due to the weight of the solution above, sufficient to balance the osmotic pressure.

A mechanical analogue of this apparatus may be helpful in understanding the nature of the operation, even though the analogy is imperfect. Suppose that several coiled springs stand on the porous partition with the piston resting upon their upper ends. If the springs were allowed to expand and raise the piston, the water would be literally pumped into the cylinder. The sugar, when in solution, is thought to have elastic properties that make possible a result similar to that accomplished by the coiled springs. The analogy is imperfect since the springs produce a pressure only in the vertical direction, whereas the pressure due to the sugar in the solution is exerted upon all of its boundaries. In the analogy we depend upon atmospheric pressure for the pumping action mentioned; but in the actual case the pressure is outward on every boundary of the water and the column cannot collapse. Under such conditions the water may transmit a tension and so the column may be forced to a height much greater than the atmosphere could lift it.

In the thistle tube there was no piston assumed, but that is not an essential part of the apparatus. The pressure is exerted within the solution and against its boundaries, and the liquid finally comes to an equilibrium position when the hydrostatic pressure at the membrane balances the osmotic pressure. Under these conditions the rates of diffusion are the same in both directions. Thus it is seen that the height of the column, when equilibrium is reached, is a measure of the osmotic pressure. When the pressures amount to several atmospheres, other and more convenient means of measuring them are employed.

150. Laws of Osmotic Action.—Extended investigations of this subject have led to some very interesting conclusions which may be stated in the form of the laws of osmotic action. We are still assuming no chemical action between solvent and solute and also that the solute is not ionized when it is dissolved.

(1) The osmotic pressure is proportional to the concentration of the solution, provided the temperature remains constant.

(2) The osmotic pressure for a given concentration is proportional to the absolute temperature.

(3) Quantities of dissolved substances which are in the ratio of their molecular weights exert equal osmotic pressures at equal temperatures.

It has been pointed out in §147 that the first of these laws is equivalent to Boyle's law for gases. The second is practically a restatement of another gas law, the law of Charles or the law of Gay-Lussac. To interpret the third it may be clearer to refer to a special case. The molecular weight of cane sugar is 342 and that of glucose is 180. If 342 grams of cane sugar are dissolved in a given volume of water and 180 grams of glucose are dissolved in an equal volume of water, the quantities are in the ratio of their molecular weights. At a given temperature these two solutions are found to exert equal osmotic pressures, which is in accordance with the third law. It should be observed that the two solutions contain equal numbers of molecules, and hence the law can be stated in the following form: Under the same conditions of temperature and osmotic pressure, equal volumes of solutions contain equal numbers of molecules. This will be recognized as the exact equivalent of Avogadro's Law. With these laws established, it is evident that the assumption made in §147 of the similarity between a dissolved substance and a gas is justified.

It has been mentioned that Boyle's Law fails to represent the relation of pressure to volume of a gas under great pressure.

There is a similar failure of the laws of osmotic pressure to represent the facts when the osmotic pressure becomes great as a result of increased concentration of the solution. The laws should be considered as applying only to dilute solutions.

We have discussed only the simplest cases of osmotic action, eliminating those involving chemical and electrical activities. With acids and salts, which ionize when put into solution, the results cannot be stated so simply, and in many instances the action is so complicated that it still remains a mystery.

151. Some Consequences of Osmotic Action.—The walls of a living plant cell are made up of cellulose lined with protoplasm, and the interior of the cell is filled with a solution of organic acids and salts with some sugar. The protoplasm is a semipermeable membrane for these substances. If the cell is placed in pure water, the water can pass through the cell walls but the dissolved substances cannot. Thus the cell is distended. If the cell is put into a concentrated salt solution, the conditions are reversed. The water that is inside the cell diffuses outward faster than the water enters the cell, and the cell shrinks in size. To illustrate these actions, cut a potato into thin slices and put some of the slices into pure water and others into a salt solution. After half an hour those that have been in the pure water will be found to be stiff and so brittle that they break easily. The others are robbed of the water which before distended the cells, and will be found as flexible as a piece of cloth.

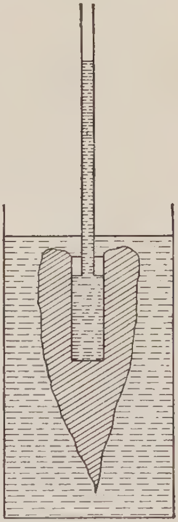


Fig 105

As a further illustration of osmotic action in its relation to plant life, consider the experiment suggested by Fig. 105. A hole about an inch in diameter is bored in a carrot, and a rubber stopper fitted to it. Cane sugar and water are put into the cavity and then the stopper is put in so as to make a tight fit. A long glass tube is fitted into the hole in the stopper, and the carrot is then immersed in pure water. The walls of the cavity act as a semi-permeable membrane; water is taken in by the cells and passed on to the sugar solution within. Thus the solution is forced up into the tube, and in the course of a day may rise six feet or more. This suggests that osmotic action is one of the agents concerned with the motion of the juices of plants and the sap in trees. Capillary action also plays an important part in these operations.

152. The Units of Mechanics.—

QUANTITY	EQUATION	ENGLISH UNIT		METRIC UNIT	
Mass		1 pound		1 gram	
Length		1 foot		1 cm	
Time		1 second		1 second	
Angle	$\theta = s/r$	1 radian		1 radian	
Area	$A = L^2$	1 foot^2		1 cm^2	
Volume	$V = L^3$	1 foot^3		1 cm^3	
Density	$d = m/v$	1 lb. per ft.^{3}		1 gm per cm^3	
Velocity	$V = s/t$	1 ft. per sec.		1 cm per sec.	
Acceleration	$a = \frac{V_1 - V_2}{t}$	1 ft. per sec.^{2}		1 cm per sec.^{2}	
Angular velocity	$\omega = \theta/t$	1 rad. per sec.		1 rad per sec.	
Angular accel.	$\alpha = \frac{\omega_1 - \omega_2}{t}$	1 rad. per sec.^{2}		1 rad per sec.^{2}	
Momentum	$M = mv$	1 lb. ft. per sec.		1 gr cm per sec.	
Angular momentum	$M = I\omega$	1 lb. ft.^{2} per sec.		1 gr cm^2 per sec.	
Moment of inertia	$I = \Sigma mr^2$	1 lb. ft.^{2}		1 gr cm^2	
		Grav.	Absolute	Grav.	Absolute
Force	$F = ma$		1 poundal		1 dyne
	$F = m$	1 pound		1 gram	
	$F = \frac{mv^2}{r}$		1 poundal		1 dyne
	$F = AHdg$		1 poundal		1 dyne
	$F = AHd$	1 pound		1 gram	
Surface tension	$T = F/L$	1 lb. per ft.		1 gr per cm	
“ “	“		1 pdl. per ft.		1 dyne per cm
Stress, pressure	$P = F/A$	1 lb. per ft.^{2}		1 gr per cm^2	
“ “	“		1 pdl. per ft.^{2}		1 dyne per cm^2
Strain	$S = l/L$		1 ft. per ft.		1 cm per cm
Coef. of elasticity	stress/strain		1 pdl. per ft.^{2}		1 dyne per cm^2
“	“	1 lb. per ft.^{2}		1 gr per cm^2	
Impulse	Imp. $= Ft.$	1 lb. sec.	1 pdl. sec.	1 gr sec.	1 dyne sec.
Torque	$T = Fl$	1 lb. ft.	1 pdl. ft.	1 gr cm	1 dyne cm
Torque	$T = I\alpha$		1 pdl. ft.		1 dyne cm
Work	$W = FS$	1 ft. lb.	1 ft. pdl.	1 gr cm	1 erg
Potential Energy	P. E. $= Mgh$		1 ft. pdl.		1 erg
	P. E. $= Mh,$	1 ft. lb.		1 gr cm	

Quantity	Equation	English Unit		Metric Unit	
		Grav.	Absolute	Grav.	Absolute
Kinetic Energy	$K.E. = \frac{Mv^2}{2}$		1 ft. pdl.		1 erg
	$K.E. = \frac{I\omega^2}{2}$		1 ft. pdl.		1 erg
Power	$P = W/t$	1 ft. lb. per sec.		1 gr cm per sec.	
"	"		1 ft. pdl. per sec.		1 erg per sec.
"	$P = W/kt$	1 H. P.			1 watt

153. Translation and Rotation.—There is an interesting analogy between the equations dealing with translation and those related to rotation. We naturally think of angular velocity as analogous to linear velocity, and of angular acceleration as analogous to linear acceleration. This does not mean that these quantities are equal or that they have the same physical nature. Distance and angle are described, one by a moving point and the other by a rotating line. For mass the analogous quantity in connection with rotation is moment of inertia, while force and torque are analogous in the sense that one produces a linear acceleration and the other produces an angular acceleration. The first table below shows the name, the symbol, and the unit of these different quantities, and the relations among them. The second table is a list of the equations which we have derived. It will be observed that any equation of the list at the right could be written down by substituting in the corresponding equation at the left the analogous term for each symbol. To illustrate from the first table we observe that, to obtain the equation that is analogous to $F = Ma$, we write torque instead of force, moment of inertia instead of mass, and angular acceleration instead of linear acceleration, and we have the familiar equation, Torque = $I\alpha$.

TRANSLATION				ROTATION		
Name	Symbol	C. G. S. Unit	Relation	Name	Symbol	C. G. S. Unit
Distance	S	1 cm	$S = \theta R$	Angle	θ	1 radian
Velocity	v	1 cm per sec.	$v = \omega R$	Ang. Vel.	ω	1 rad. per sec.
Acceleration	a	1 cm per sec.^{2}	$a = \alpha R$	Ang. Acc.	α	1 rad. per sec.^{2}
Mass	m	1 gram	$\Sigma mr^2 = I$	Mom. In.	I	1 gram cm^2
Force	F	1 dyne	$Fl = T$	Torque	T	1 dyne cm

EQUATIONS

$S = vt$	$\theta = \omega t$
$v = v_o + at$	$\omega = \omega_o + \alpha t$
$S = v_o t + \frac{1}{2} at^2$	$\theta = \omega_o t + \frac{1}{2} \alpha t^2$
$v^2 = v_o^2 + 2aS$	$\omega^2 = \omega_o^2 + 2\alpha\theta$
$F = ma$	$T = I\alpha$
$F = \dfrac{mv - (mv)_o}{t}$	$T = \dfrac{I\omega - (I\omega)_o}{t}$
$Ft = mv$	$Tt = I\omega$
$W = FS$	$W = T\theta$
$P = Fv$	$P = T\omega$
$\text{K.E.} = \frac{1}{2} mv^2$	$\text{K.E.} = \frac{1}{2} I\omega^2$
$\text{Mom.} = mv$	$\text{Ang. Mom.} = I\omega$

SOUND

CHAPTER XI

WAVE MOTION

154. Introduction.—Study of the phenomena of sound has shown that every sound perceived by the ear has originated in a vibrating body or system, has been propagated through an intervening material medium as a wave disturbance, and finally has been received by the ear and registered as a sound sensation. This text will not concern itself with this last phenomenon, though it is one whose explanation requires the application of the principles of physics as well as those of several allied sciences. It will consider the vibrating bodies and vibrating systems which are the sources of sounds, and the characteristics of the wave motion by which energy from these sources is propagated outward through the surrounding medium, whether it be a solid, a liquid or a gas. Opportunity will be given to discuss other waves as well as those of sound, since the propagation of energy by waves plays such an important part in natural phenomena; for at no other point in the subject can the complicated phenomena of wave propagation be so well illustrated as in sound, where the waves are in material media and readily lend themselves to direct experimental demonstration. The ideas to be developed at this time in sound are such as may be carried over very directly into other subjects. The place to begin the study of radio or X-rays is in mechanics and sound.

That bodies or systems which are the sources of sounds owe this property to the fact that they are vibrating may be demonstrated by experiment. A sounding tuning fork, the motion of whose prongs is not evident to the eye, may be shown to be in motion by dipping the ends of its prongs into water. The outflung water is sufficient evidence of this motion. Better experimental devices show that this motion is quite accurately simple harmonic. That metallic rods and bells which emit a sound as a result of a blow are in vibration is evident when a small ball suspended by a fine thread is held lightly against them at the right points. In all the above instances the vibrations would also have been evident through the sense of touch. By placing a finger against any of these bodies when they are sounding, one is able to feel the rapid vibrations. The vibrations of the pipes of lowest pitch in large organs are often felt by the body at the same time that they are perceived through the sense of hearing.

155. Vibrating Bodies.—A vibrating body repeats the same series of states at regular time intervals. The motion included in one of these cycles is called a vibration, and the time required to repeat a cycle is the period of the vibration, T. The number of vibrations per second, n, is the reciprocal of the time of one vibration, $n = \frac{1}{T}\cdot$ The maximum displacement in any vibration from the position of equilibrium measures the amplitude of the vibration.

An illustration of the simplest kind of vibration is the motion of a child's swing. The motion of the swing as it goes over and back constitutes a complete vibration. Vibrations are in general more complex than this, but however complex, the complete motion repeats itself during each vibration.

156. Wave Motion.—A vibrating body in contact with a solid or immersed in a liquid or gas gives up its energy continually. This energy is carried away from the vibrating body by the waves which its vibrations set up in the contiguous medium. The energy of vibration given to quiet water where a stone is dropped into it is rapidly carried away by a number of circular waves which spread out on the surface of the water. A number of the characteristics of wave motion in general may be recognized in these simple water waves. The waves consist of a series of alternate elevations and depressions which travel out from the center with a constant velocity in all directions. The elevations are called crests, the depressions, troughs.

That the water in a crest does not travel outward is evident when it is noticed that small floating and suspended objects appear simply to rise and fall as crest and trough pass them. The water particles have similar motions. On the supposition that in a uniform medium adjacent particles are equidistant, there will be a constant phase difference between the vibrations of the adjacent particles which lie in any line of propagation of the wave. It is by handing on energy from one vibrating particle to the next that energy is transmitted by waves.

157. Wave Length.—The distance from crest to adjacent crest with the intervening trough is called a *wave length*. This length is definitely connected with the velocity with which the wave travels, since it is the distance traveled by the wave while the individual water particles are making one of their vibrations. Thus $\lambda = vT$ where λ is the wave length, v the velocity of propagation of the wave and T the period of vibration of the water particles.

158. Transverse Waves.—If the vibratory paths of the particles of the medium are normal to the direction in which the wave travels, the wave is said to be a transverse wave. This term is also applied to waves where the motion of the particle is not rectilinear, provided it has a considerable component at right angles to the direction of propagation. For example, the ripples that travel over the surface of quiet water are considered as transverse waves, although the motion of the particles of water near the surface is approximately circular.

159. Wave Fronts.—As previously stated, there is a constant phase difference between the vibrations of equidistant particles in any line of the wave propagation. This fact may be used to define certain properties of waves. Those particles of the medium whose distances from the source are such that they are starting their vibrations for the first time and are accordingly at the zero phase of their vibratory motions, lie in a surface which is called a wave front. More generally, any surface in which all the particles lying in it are, at any moment, in the same phase of their vibratory motion, is also called a wave front. In the case of a uniform medium where the waves travel out with equal velocities in all directions, this wave front is spherical in form when the source is small. In the case of a pebble dropped in a pool of water the wave front is circular. When a flat surface of sufficient size is made to move back and forth periodically in a direction normal to its face, waves will be formed in the medium in which it is immersed and these waves will have a plane wave front.

160. Wave Length.—The definitions of wave length so far given, *i.e.*, the distance between two adjacent crests or the distance traveled by the wave during a single period of vibration of a particle of the transmitting medium, may be supplemented by a third. A wave length is the distance, along the line of propagation, between the nearest particles in the same phase of their vibratory motions. A consideration of these three definitions shows that the first and third are derived from the second, which may be taken therefore as the general definition of wave length.

161. Rays or Direction of Propagation.—Starting anywhere in a wave front, if a point moves so as to keep in the advancing wave front and to have its path always normal to the wave front, it will trace out a line which is called a ray or, more at length, the line of propagation of the wave through this portion of the medium. In the case of the water wave these paths are all radii of the circular wave fronts. In the case of the plane waves these lines, everywhere normal to the wave fronts, are parallel to each other. Each ray may be considered to be a path along which any given portion of the wave energy is being transmitted. Where from lack of uniformity of the medium, the wave fronts become distorted, assuming complex forms, the rays traced as described above become correspondingly curved, still representing the lines of flow of energy. There are cases where the direction of propagation is not normal to the wave front, but these cases will not be discussed here. Assuming the paths of flow of energy to be in general normal to the wave fronts, it will be possible to change the direction of flow of energy by changing the direction or the shape of the wave front. Any device which will change the spherical wave front of waves diverging from a point into a plane wave front will change the lines of flow of energy from the radii to parallel lines. When energy is flowing along parallel lines it is being kept together, as light in a searchlight beam is conserved. Should the diverging spherical wave front be changed into a spherical converging front, the energy would then flow along radii, not outward from a center but inward toward a center. In this case there is a concentration of energy. Devices by which the forms of the wave fronts are changed are very important and will be more thoroughly treated later.

162. Sound Waves.—Sound waves are compressional waves. A single wave consists of a region of compression and a region of rarefaction. In the region of compression the density of the medium is greater than normal; in the region of rarefaction, less than normal. To transmit such waves, the particles of the medium are thrown into vibration in turn, as the wave travels outward from the source. These vibrations are rectilinear, their direction being parallel to the direction in which the wave is propagated. Consider a tuning fork which vibrates by the alternate separation and approach of its two prongs. When the prongs separate, the air in contact with the outer sides of the prongs becomes compressed, that between the prongs rarefied. These regions of compression are relieved by the air particles contained within them moving in all directions except, naturally, toward the sides of the prongs. This outward motion of the particles from the first regions of compression results in compressions of the surrounding region, which in turn are relieved by their particles moving outward. A compression is thus transmitted outward from each prong by the air particles taking up this outward motion in turn. When the prongs approach one another the air in contact with the outer sides of the prongs is left less dense, *i.e.*, regions of rarefaction are formed. Air from the surrounding regions moves into these spaces, leaving regions of rarefaction behind. These are relieved by air from the next outer layers flowing inward. Thus a region of rarefaction is propagated outward from each prong by having the air particles move inward in turn toward the vibrating fork. It is evident that the individual air particles have transmitted outward these regions of condensation and rarefaction by simply moving out and then back, *i.e.*, by making a vibration in the line of propagation. Thus, as the fork continues to vibrate, its energy is carried away by a train of compressional waves, each wave of which consists of a region of compression in which the particles are moving outward in their vibratory paths, and a region of rarefaction in which the particles in vibration are moving backward toward the source.

163. Longitudinal Waves.—Waves of the above type in which the vibrations of the particles of the transmitting medium are linear and in the direction of wave travel, are called longitudinal waves.

164. Wave and Vibratory Motions.—The velocity which these waves possess depends upon the rapidity with which such changes in density can be handed on from point to point in the medium. The velocities possessed by the particles in their vibratory motions are quite independent of the velocity of the wave. Should the fork be making 250 vibrations per second, each of the air particles of the medium is also vibrating back and forth along the line of wave travel 250 times per second. The amplitude of the vibrations of the air particles in thus handing on the energy is but a few hundredths of a mm, and their maximum vibrational velocity is but a few cm per second. That the above is essentially a true picture of the propagation of a sound wave in a gas will have numerous demonstrations before the subject is finished. The phenomena in liquid and solid media are similar; *i.e.*, sound waves, whatever the medium of transmission, are longitudinal waves consisting of alternate compressions and rarefactions.

165. Intensity of Sound.—A sound is said to possess an intensity at each point of a medium through which it passes.

This property of sound is measured at any point by the amount of sound energy which passes per second through one square centimeter of surface taken normally to the direction of propagation at the point. The energy per second which thus passes through any imaginary surface determines the energy of vibration of the particles of the medium lying in the surface. As this in turn is proportional both to the square of the amplitude of the vibrations of the particles and to the square of the number of vibrations per second, the intensity at the point is proportional to this product, a^2n^2. This follows, if the vibrations are all simple harmonic, from the expression for the maximum value of the kinetic energy of such vibrations. $E = \frac{1}{2}m\omega^2a^2$ where $\omega = 2\pi n$, $\mathrm{E} = \frac{1}{2}m(2\pi n)^2a^2$. Therefore E is proportional to n^2a^2. These relations hold equally well for other kinds of waves. Thus the transverse waves shown in Fig. 106[1] possess intensities determined by their amplitudes and frequencies. If the wave *a* has an intensity represented by I, the wave *b* has an intensity of $9I$. This is because the frequency of *b* is 10 times that of *a*, which alone would give *b* an intensity of $100I$; but an amplitude of .3 for *b* reduces the intensity of *b* to $.3^2$ or .09 that of *a*. Together, they produce $(100 \times .09)\ I$ or $9I$. In *c*, which is the combination of *a* and *b*, the intensity is $10I$, or the sum of the intensities of the component waves.

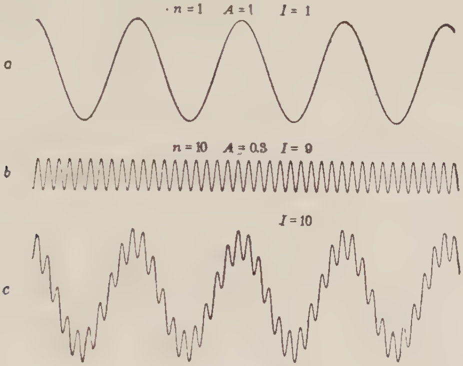


FIG. 106

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[1] From *The Science of Musical Sounds*, by D. C. Miller.

166. Effect of Distance Upon Intensity.—In a uniform medium sound waves are spherical when the source is small. If the medium is non-absorbing, the sound energy passing through any two spherical surfaces, concentric with the sound waves, will be the same. The areas of these surfaces are $4\pi r_1^2$ and $4\pi r^2_2$, where r_1 and r_2 are the radii of the inner and outer spheres respectively. Since the intensity of sound I, by definition, is the energy passing per second through each unit of area normal to the direction of propagation, the energy passing through the inner sphere per second is $4\pi r_1^2 I_1$ and through the outer sphere is $4\pi r^2_2 I_2$. Thus, $4\pi r_1^2 I_1 = 4\pi r^2_2 I_2$, and

$$\frac{I_1}{I_2} = \frac{r^2_2}{r_1^2},$$

or the intensity of the sound varies inversely as the square of the distance from the source. Actual intensities seldom follow this inverse square law since the usual sound medium is not uniform but rather contains many objects which absorb and reflect.

167. The Effect of the Source Upon Intensity.—The intensity at any point depends upon the rate at which the vibratory source gives out energy in the form of sound. While this rate increases with the energy in the vibrating source, the relation is not one of direct proportionality, but is determined by the conditions under which the transfer of energy occurs. Sources of high frequency lose a larger fraction of their energy per second than do similar sources of low frequency.

The shape of the vibrating body is also a factor in the rate at which it loses energy to a medium in which it is immersed. Thus a vibrating string produces sound of small intensity since the air readily flows around the string as it moves, and weak condensations and rarefactions result from its vibrations. The same string fastened to a suitable surface, *i.e.*, a sounding board, causes the surface to vibrate. This through its large vibrating area produces greater changes in density in the air, and accordingly vibrations of greater amplitude and waves of greater energy are formed.

168. Loudness.—While loudness and intensity are related, the relation is not a simple one, since the loudness of a sound depends not only upon its intensity but also upon the physical characteristics of the ear receiving it.

169. Pitch.—The frequency of vibration of the source which determines the number of waves sent out per second not only affects the intensity of the sound, as indicated in §165, but determines also another property of sound, *i.e.*, its pitch, which is measured directly by the number of vibrations per second. As the number of vibrations increases, the pitch is said to become higher. The ear is usually sensitive to sounds whose frequencies of vibration range between 20 and 20,000, though some ears extend this range materially. Waves of still higher frequency, though inaudible, are beginning to be used and are spoken of as high frequency waves.

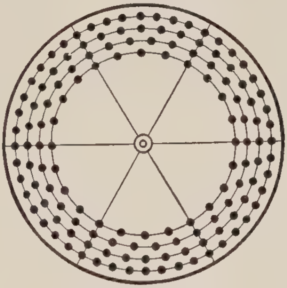


Fig. 107

A simple device for demonstrating the nature of pitch is the siren shown in Fig. 107. It consists of a rotating circular plate provided with a number of concentric rings of equidistant holes. When a current of air is directed by means of a pipe through the holes of any one of the circles, the succession of puffs thus produced results in a note of a definite pitch being formed. Its pitch rises or falls as the air is directed to circles of a greater or a smaller number of holes. Similarly the pitch rises or falls as the speed of rotation increases or decreases while a single circle is used.

In many cases it is possible to attach to sounding bodies devices which register the number of vibrations per second. Thus a bristle fastened to the prong of a sounding tuning fork may register the motions of the prong upon a moving smoked plate, tracing out a sine curve, the number of waves per second equaling the number of vibrations per second.

170. Quality.—A third characteristic property of sound is its quality. This property, together with the other two just discussed, distinguishes one sound from another. It must be sufficient here to say that the quality of sounds arises from the fact that the vibrations causing them are usually quite complex and that the resulting sound waves have the same degree of complexity. That sounds of the same pitch and intensity—for example, those produced by two persons singing the same note with equal loudness—are readily distinguishable, is a familiar fact. This difference of quality arising from complexity of the waves will be treated more at length later.

171. Velocity of Sound.—Measurement has shown that sounds of all pitches travel with the same velocity in the same medium. The same is true of sounds of different intensity, provided the intensity is not too great, as might be the case in an explosion. In such a case, the velocity would be greater. The velocity of sound at 0° C. varies greatly in different media, being 330.36 meters per second in air, 1,435 meters in water, and 4,975 meters in steel. Theory shows that the velocity of sound waves should be

$$V=\sqrt{\frac{e}{d}},$$

where e is the coefficient of volume elasticity, and d the density. For a gas there are two principal coefficients of volume elasticity (§98). When the gas, during the change of volume, remains at a constant temperature, the coefficient is called the isothermal coefficient, and is here written e_i. This coefficient is numerically equal to the pressure (§124). When the change in volume of the gas occurs under such conditions that there is no flow of heat into or out of the gas during the change in volume, then the coefficient obtained is the adiabatic coefficient, e_a. This coefficient is different for different gases, but is always larger than the isothermal coefficient.

Sound waves have been assumed to be compression waves consisting of condensations and rarefactions. Any portion of the air through which a wave is passing is alternately compressed and expanded. This means that it is alternately warmed and cooled, since the sound waves have such a high frequency that the time of compression or expansion is too short for an equalization of temperatures with the surroundings to take place. Under these conditions, which are called adiabatic, the adiabatic coefficient of elasticity, e_a, should be used. In the case of air it is about 1.41 times larger than the isothermal coefficient. Since, as is shown in §124, $e_i = p$ and $e_a = 1.41e_i$ or $1.41p$ for air, the velocity for sound in air is $V = \sqrt{\frac{1.41p}{d}}$, a result which gives values in close agreement with experiment. In the case of liquids and solids, there is not this local change in temperature, since the compressions are much smaller and the chances for equalization of changes through thermal conduction are much better. The formula $V = \sqrt{\frac{e_i}{d}}$ therefore applies for liquids and solids.

The air is subject to changes of barometric pressure, but since the density varies also directly as this pressure, the velocity $V = \sqrt{\frac{e}{d}} = \sqrt{\frac{1.41p}{d}}$, remains constant. A general change in the temperature of the air changes its density but not its pressure. The velocity therefore varies with the temperature. It increases approximately 0.6 meters per second per centigrade degree of temperature change.

172. The Doppler Effect.—A passenger on a through train often passes small villages at high speeds. If he is observant he will notice that the automatic gongs ringing at crossings appear to experience a change of pitch as they are passed. There is a distinct lowering in the pitch due to the fact that upon approach the pitch heard is higher than the true pitch, while it is lower upon recession. This effect is known as the Doppler effect, and is larger, the greater the speed of the train.

The change in pitch is readily computed. The time required for a single sound wave to pass a stationary point is T, the period of the vibrating source. $\frac{\lambda}{V} = T$, where λ is wave length of sound and V is the velocity of sound. The number of waves passing per second equals

$$n = \frac{1}{\mathrm{T}} = \frac{V}{\lambda}.$$

If the speed of the train is v, the relative speed with which passenger and wave pass each other is $V + v$, and the time required for passing will be T'.

$$\mathrm{T}' = \frac{\lambda}{V + v}.$$

The number of waves passing per second in this case will be n'.

$$n' = \frac{1}{T'} = \frac{V + v}{\lambda}.$$

Since the pitch of the source, under these circumstances, would be judged by the number of waves passing per second, n' is the apparent pitch of the gongs. Its relation to the real pitch is given by the expression

$$\frac{n'}{n} = \frac{V + v}{V}, \text{ or } n' = n\frac{V + v}{V}.$$

As $\frac{V + v}{V}$ is larger than unity, the apparent pitch is higher than the true pitch. It can be shown in a similar manner that the observed pitch is lower than the true pitch upon recession.

The student should show what results are obtained if the observer is stationary and the source of sound is moving, as when the train whistles continuously while passing a station.

173. Reflection of Sound.—The familiar echo is the result of the reflection of sound waves from a surface which is sufficiently regular so as not to break up the wave forms during the process of reflection. Under favorable conditions, waves on the surface of water may be seen to be thrown back from some surface against which they strike, the reflected waves having the same wave length as the original waves. If the surface is uneven, as in the case of a rocky shore where the rocks have sizes comparable with the wave lengths, the waves upon reflection are broken up into many waves of irregular form. This same phenomenon occurs in sound under similar circumstances, and no echo results. The return of a sound wave to its source, which is commonly called an echo, finds use, by methods developed during the war, in determining sea depths. Sound waves are directed downward from a traveling ship and the time required for the reflected sound or echo to return measures the depth, since $s = vt$, where t is half the time observed. This is a much quicker and more satisfactory method of measuring ocean depths than the method of the plumb line, and does not require that the vessel reduce its speed.

Reflection of sound waves occurs whenever sound passes from one transmitting medium to another. The reflection may be only partial, the energy being passed partly through the surface of separation. The English physicist Tyndall found that such reflections occurred between vertical air columns of different densities to such an extent that sounds from the fog-horns of lighthouses often could be heard but short distances out at sea, the sounds having been returned to the coast by reflection. Sound signals are therefore transmitted under water with greater certainty than through the air. With the development of the technique of measuring reflected sounds, there is the possibility, now partly realized, of examining the structure of the earth by registering the time required for the reflection of sounds from the various strata. The walls of auditoriums occasionally so distribute sounds by reflection that speech from the stage cannot be understood. The study of the acoustic properties of auditoriums is a matter of importance. Connected with this is the study of the absorbing, reflecting, and transmitting properties of the materials which may be used on the walls.

174. Reverberation.—When a sound is produced within walls, as in living rooms or in auditoriums, often a hundred or more reflections occur before the intensity of the sound falls below audibility. Under these circumstances, the direct waves from the source which reach the auditor first are followed by waves reflected in turn from surfaces at varying distances, so that the intensity of sound from a continuous source will grow until a maximum has been attained. Likewise, when the source stops vibrating, silence is not immediate but comes only after the waves, reflected between the various surfaces of the room, become too weak to be heard. This general diffusion of sound through a room due to multiple reflections is called reverberation. The period of reverberation of a room is the time required for the sound, arising from a source of standard intensity, to decay to the limits of audibility. A standard intensity source is one giving a sound whose intensity is a million times greater than a sound at the limits of audibility. The rise in intensity of a continuous sound in a room is illustrated in Fig. 108a, where the ordinate is the average intensity I, and the abscissa is the time t. When the source stops vibrating, as at time B, there is a very rapid decrease in intensity. Fig. 108b shows the necessity of sufficient rapidity of decay if succeeding sounds are to be heard distinctly.

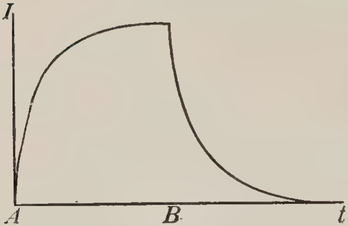


Fig. 108a

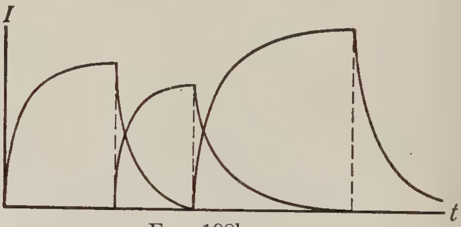


Fig. 108b

175. Coefficient of Absorption.—That the intensity of sound does not increase indefinitely when a continuous source sounds within a room is due to the fact that the sound loses some of its energy at each reflection because of the absorption and transmission of the reflecting surfaces. The maximum intensity is reached when an equilibrium is attained; that is, when the gain of energy from the source just equals the loss through absorption and transmission. At a perfect reflector the reflected energy equals the incident energy; at a perfect absorber the reflected energy is zero. Hard smooth surfaces reflect the sound almost completely, whereas soft rough surfaces absorb highly and reflect very little. Most materials which compose the surfaces of rooms partially absorb and partially reflect the sound which falls upon them. The only surface which acts as a perfect absorber is an opening, such as an open window, since all of the sound energy incident upon this surface passes through and is lost to the room. For present purposes, loss of energy by absorption and transmission will be treated as if it were by absorption alone.

Since the period of reverberation of a given room is the time required for the sound to decay to inaudibility, when a source of standard intensity has been sounding, it is evident that an increase in the absorbing qualities of the reflecting surfaces will shorten this period, since this change will increase the rate of decay. If one surface is replaced by another and the period of reverberation of the room is unchanged, the two surfaces must have the same absorption. Use is made of this principle to determine the coefficients of absorption of materials. As a standard of absorption, an open window of an area of 1 square meter is used. Should a surface of 4 square meters area covered by a given material give the room the same reverberation period as the open window, then 4 square meters of this material has the same absorption as 1 square meter of open window. One square meter of the material will therefore absorb .25 as much as the unit window. Its coefficient of absorption is said to be .25. The expression coefficient of absorption accordingly means the fraction of the incident energy which is absorbed. The following table gives the coefficients of absorptions of various materials usually found in rooms and auditoriums, as measured by Sabine.

COEFFICIENTS OF ABSORPTION

Open window	1.000
Wood sheathing (hard pine)	.061
Plaster on lath	.034
Plaster on tile	.025
Brick set in Portland cement	.025
Carpet rugs	.20
Hair felt (2.5 cm thick, 8 cm from wall)	.78
Audience (per person)	.44
Wooden seats (per seat)	.003
Upholstered seats (per seat)	.300
Glass	.027

176. The Absorbing Power of a Room.—From the table it is seen that the coefficient of absorption for plaster on lath is .034. This means that 700 square meters of plastered surface would have the absorbing power of 23.8 square meters of open window. Similarly, 300 square meters of carpet rugs would correspond to 60 square meters of open window. In general terms, the product of the coefficient of absorption, a, and an area, s, gives the absorbing power of that area. The total absorbing power, A, of all the surfaces which are found in the room will be the sum of the absorbing powers of the separate surfaces, or

$$A = a_1s_1 + a_2s_2 + a_3s_3 + \text{etc.};$$

or $A = as$, where a is the mean coefficient of absorption for the entire room, and s is the total area.

177. The Acoustics of Auditoriums.—It is a well-known fact that auditoriums differ greatly in their acoustic properties. Occasionally there are large surfaces in the walls or ceilings, of high reflecting power and of such shape as to bring intense sound beams to focus on limited areas of the auditorium floor. When the paths traveled by these beams are considerably longer than those of the sounds coming directly from the stage, strong echoes are heard. Fig. 109 shows an instance of this kind where a concave area of the wall focuses a beam on a small area of the gallery. A person sitting there would hear a speaker on the stage by a direct beam and also, a little later, by the beam which is reflected and focused. Under such circumstances confusion would inevitably result. Imperfections of this kind can be remedied by covering such reflecting surfaces with absorbing materials, or by changing their shapes.

In addition to echoes, large auditoriums often have too long periods of reverberation. Sabine developed the following formula for the period of reverberation:

$$t = .164\frac{V}{A},$$

where V is the volume of the auditorium in cubic meters and A

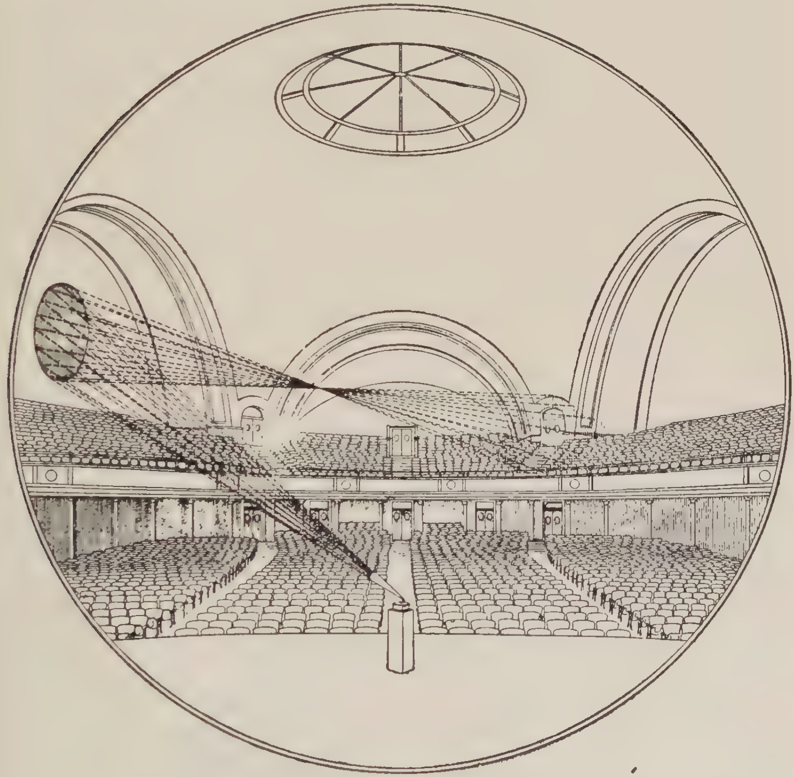


FIG. 109 *By permission of F. R. Watson*

is the total absorbing power of the room. If all the dimensions of a given room increase in the same proportion, it will be evident that the value of t will become greater, since V increases with the cube of the dimensions and A with the square only. Thus auditoriums are much more likely to give trouble through reverberation than small rooms. Experience has shown that a reverberation period of from one to two seconds, depending upon the size of the auditorium, is best for speech, and that a somewhat longer period is better for music. For a room of a given volume, it is evident that the period of reverberation can be altered by changing A, which is easily done by varying the absorbing powers of some of the reflecting surfaces. Since a sound persists in the ear for about 1/15 of a second, those surfaces which will reflect directly to the audience within 1/20 of a second are generally made good reflectors to give intensity to the sound, while those surfaces farther away are given the necessary absorbing power to give the desired period of reverberation.

The way in which a sound becomes rapidly distributed throughout an auditorium is illustrated by the sound pictures of Sabine after the Foley Method. In Figs. 110 and 111, a cross-section of a theater in miniature is shown. A simple wave, starting from the stage in the first photograph, is shown in the second, reduced to a very short arc in the uppermost corner of the second gallery, while most of the wave by reflections and re-reflections at different surfaces is well distributed throughout the auditorium. These waves, started by an electric spark, are registered upon a photographic plate placed below the model by a second spark suitably situated above and properly timed. The outline of the model is determined by its shadow, while the sound waves themselves refract the light sufficiently to form sharp images on the plate.

178. Huygens' Principle.—With each particle of the medium vibrating with the same frequency and the same kind of vibrations as the vibrating source, however complex these may be, the question may arise as to why each particle of a medium through which waves are passing should not be regarded as a source from which waves are sent out. If the vibrating particles, a, b, c, etc., in the wave front AB in Fig. 112 are regarded as centers from which waves are sent out just as from the original source C, these waves, which may be called wavelets,

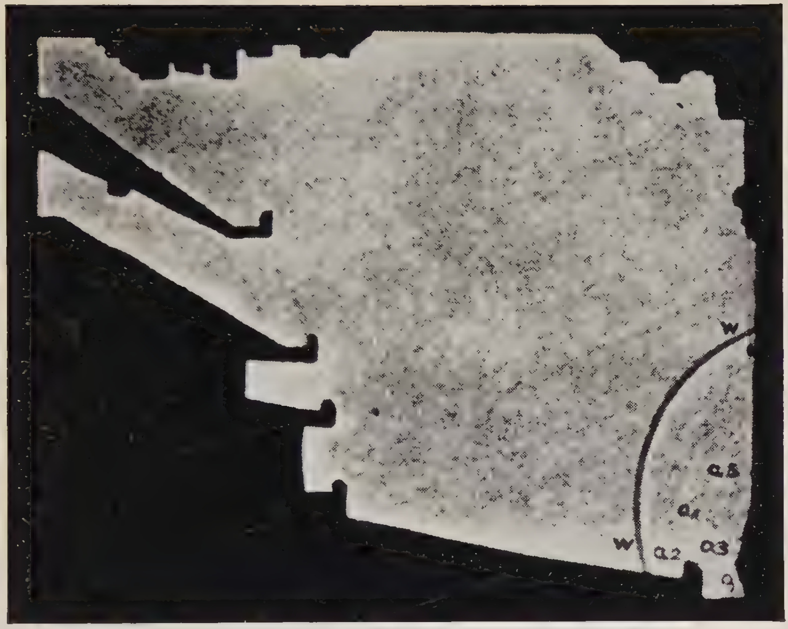


Fig. 110

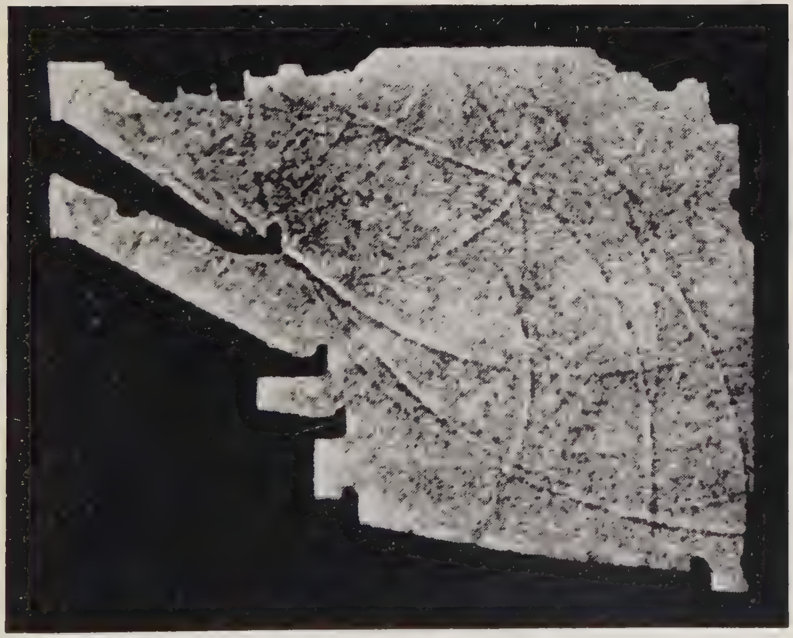


Fig. 111

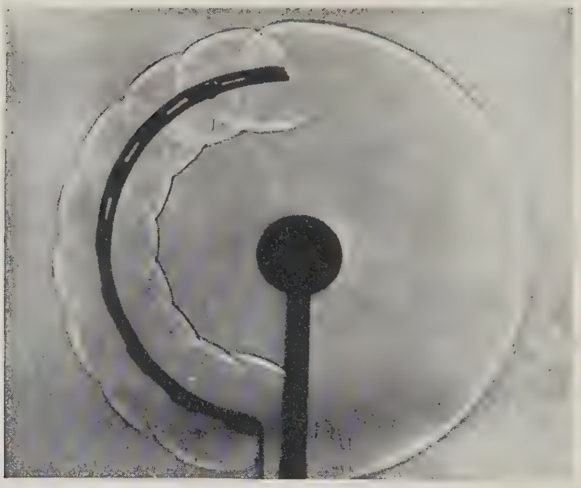


Fig. 113

traveling with the same velocity as the primary wave AB, are seen to have their wave fronts after a short time at the places indicated by the small arcs. Keeping in mind that all of the particles lying between a and b are the centers of similar arcs, it will be seen that all these wavelets will unite on the large circle $A'B'$, which may be regarded as a new wave front. The process is then repeated with the formation of the subsequent wave fronts. It can be shown that these secondary waves react with one another in such a way that they transmit energy in uniform media almost wholly in the direction normal to the primary wave fronts; that is, the energy of the vibrating particle a is transmitted by the secondary wave quite completely to a very limited portion of the wave front at a'. It is for this reason that complete circles are not drawn about a, b, c, etc., as about C. It explains also why energy is not propagated backward to the source C from the vibrating particles of the medium. This idea that each particle of a medium through which a wave is passing becomes a wave source is called Huygens' Principle.

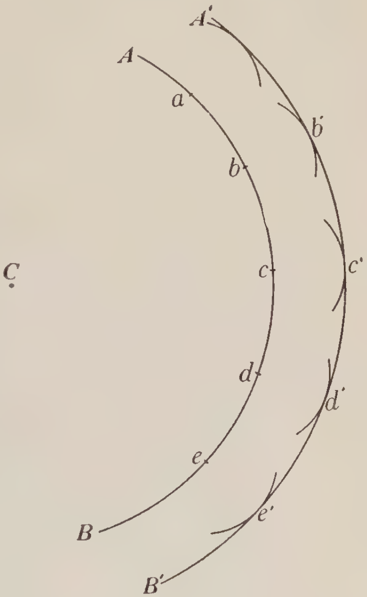


Fig. 112

In Fig. 113 a sound wave has been partially passed through a series of small apertures in a screen concentric with the wave. To the extent to which each small aperture may be regarded as a point, it becomes the center or source of an independent wave, in accordance with Huygens' Principle. The new wave front beyond the screen is the envelope of these wavelets and, with that portion of the wave unaffected by the screen, will form the complete wave, undistorted in form. Each point on this screen between apertures acts as a Huygens source which throws back energy by reflection, and these small surfaces are thus seen to be the centers of a series of reflected wavelets whose envelope forms a reflected wave front. This sound photograph is the work of Professor Foley of Indiana University.

179. Reflection of Spherical Waves.—Huygens' Principle readily explains the act of reflection. When a spherical wave, as *AB* diverging from a source *C*, in Fig. 114, strikes a flat surface *EF* at which it is reflected, its form and direction of travel after reflection are found from the following considerations. If the surface *EF* were not present, the wave *AB* would advance to the position *A′egfB′*. *AB* first strikes the surface at *v*, which point becomes the center of a reflected wavelet. Reflection means that a wavelet, instead of traveling outward, becoming the arc shown at *g*, is thrown backward toward *C* an equal distance, forming arc *g′*. That portion of the wave incident at *u* makes *u* a center of a wavelet which travels backward to *h′* instead of to *h;* similarly for all points between. These wavelets unite along the wave front *eh′g′i′f*. The construction of the figure results in this reflected wave front having the same curvature as the incident wave *A′egfB′*; its center is, however, at *C′* as far back of the reflecting surface as *C* is in front, and the line joining them is normal to the reflecting surface. A train of these reflected waves would make the same impression on the ear as if the source had been at *C′* instead of at *C*. An echo seems to come from a point as far back of the reflecting surface as the source is in front. A second Foley sound photograph, Fig. 115, shows how fully the application of Huygens' Principle gives results agreeing with experiment.

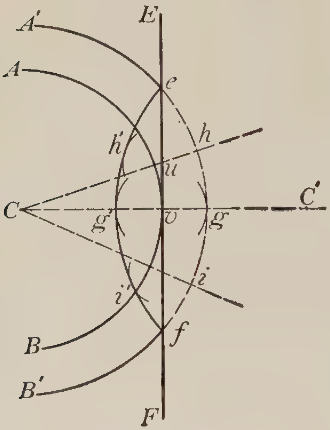


Fig. 114

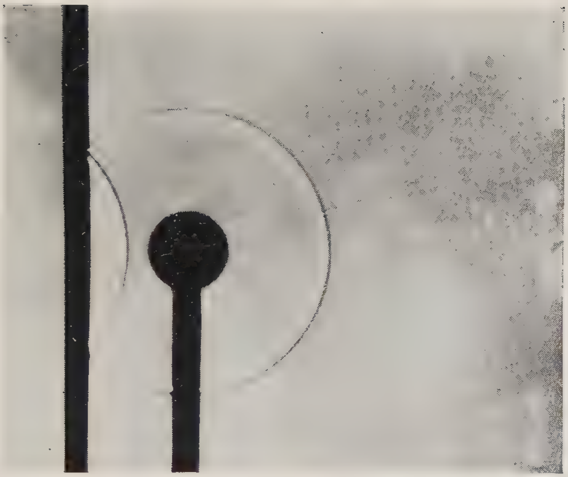


Fig. 115

180. The Reflection of a Plane Wave at a Plane Surface.—The plane wave AB, Fig. 116, is first incident on the plane reflecting surface EF at the point A. If the reflecting surface EF were not present, the disturbance incident at A would have traveled to A' in the time the disturbance at B takes to reach B', forming a new wave front $A'B'$ parallel to AB. The effect of EF is to cause the disturbance incident at A to arrive somewhere on the arc g whose radius equals AA' by the time B reaches B'. Similarly, the disturbance incident at C would have gone to C', had no reflecting surface intervened. It does travel to the arc c where $Cc = CC'$. The disturbance at B is just incident at B' as the disturbances of the incident wave at A and C have extended to the arcs g and c after reflection. As the successive points of the surface EF from A to C to B' become in turn the centers of wavelets, it is evident that these wavelets will all unite along the plane wave front gcB', which is accordingly the reflected wave front. The lines of energy flow are normal to the wave front after as well as before reflection. The line uA, normal to the wave front AB, is the path of travel before reflection of the energy incident at A; while Ag,

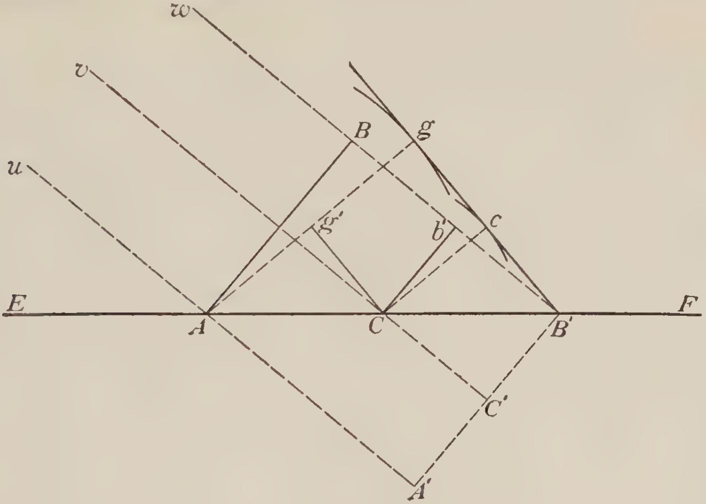


Fig. 116

normal to gcB', is the path after reflection. At the instant that part of the wave front AB which travels along vC is incident at C, that incident at A has gone to g' and the disturbance of B to b', thus making a wave front $g'Cb'$. Cg', the reflected portion, moves toward gc, and Cb' toward oB'. The angle BAB', called the angle of incidence, is seen by the construction to be equal to the angle $AB'g$, the angle of reflection. $b'CB'$ also equals $g'CA$.

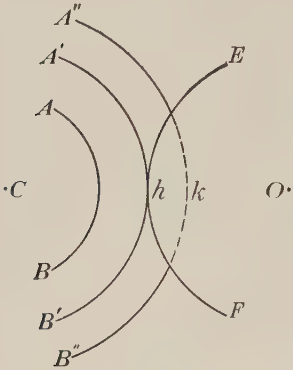


Fig. 117a

Problem 1.—Spherical waves with wave fronts AB, $A'B'$, $A''B''$, Fig. 117a, diverge from a source C and are incident upon a spherical surface whose center is O. If this reflecting surface were not present, the wave front $A''KB''$, would have been formed. Using Huygens' Principle, locate the reflected portion of the wave front $A''B''$. Compare its curvature with that of the incident wave.

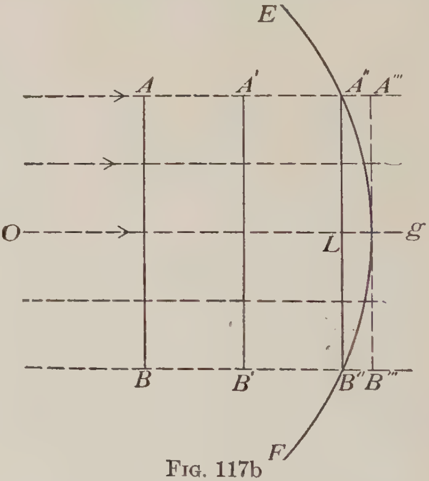


Fig. 117b

Problem 2.—Plane waves AB, $A'B'$, etc., are incident upon a concave reflecting surface EF whose center is at O, in Fig. 117b. If it were not present, the wave front $A''B''$ would travel to the position $A'''B'''$. Using Huygens' Principle, determine the reflected wave front at instant the point L in $A''B''$ is incident at the mirror.

Problem 3.—In Fig. 117c, plane waves AB, $A'B'$, $A''B''$ are incident upon a surface EF, having a minute opening at its center. Using the Principle of Huygens, construct the wave fronts of the waves passing through the aperture.

Problem 4.—Show that the reflected wave front in Fig. 117c is a plane wave front.

181. The Interference of Sound Waves. Beats.—One characteristic of the wave propagation of energy is that any number of waves may be passing through the same medium at the same time, yet each wave travels as if it alone were present. In the case of sound waves where the waves are transmitted by the vibrations of the particles of the medium, it will be evident that when more than one wave is present the particles must have impressed upon them several vibrations at once. Their motions will not therefore be of the simple kind, so far described, which are sufficient to transmit single waves.

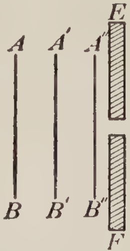


FIG. 117c

If there are two similar tuning forks side by side vibrating continuously, one 100 times per second and the other 101, it is evident that if at some instant they are both moving in a similar way, *e.g.*, both beginning to separate their prongs, each is producing a condensation in the regions in contact with the outside of the prongs. If the disturbance is not too intense, the actual displacement of the particles at any point distant from the two sources under these two influences will be the sum of what each would have produced had it acted alone. If the forks are similar and vibrating with the same amplitudes, these displacements are equal, and the sum double that of either. Since, when one makes 100 vibrations, the other makes 101 vibrations, it is evident that when the first has made 50 vibrations the other will have made 50½ vibrations. This means that when the first begins to separate at the beginning of its 51st vibration, the second will begin to draw together, since it is one-half vibration ahead of the first. The second fork therefore is producing a rarefaction, and the outward propagation of a rarefaction requires the particles of the medium to be moving back toward the source in their vibratory motions. Since the first fork, sending out a condensation, requires them to move outward, it will be evident that the particles, being required to move in opposite directions at the same time, can only move an amount equal to the difference of the displacement required by the condensation and rarefaction individually. If these are equal, there will be no motion. When waves react upon each other in such a way that the individual displacements of the air particles are added or subtracted, the waves are said to interfere.

In the first case where the effect is additive, the amplitude being double (the effects being assumed equal), the intensity which varies as the square of the amplitude is 4 times that of either one alone. In the second case the intensity is zero. After another 50 vibrations of the first fork, the second fork has gained a second half vibration, and the two forks will again be in the same phase of vibration and there will again be the

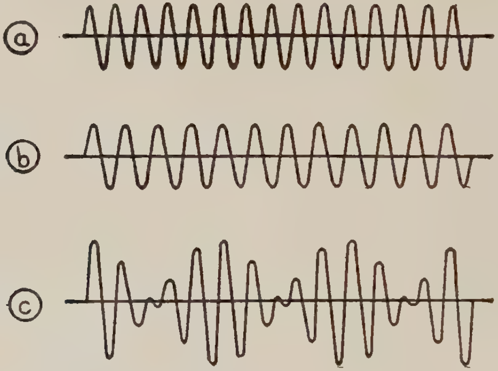


Fig. 118

additive effect. Persons listening to the forks will hear accordingly a note which in its intensity rises to a maximum when the forks are in the same phase and falls to zero when they are in opposite phase. These variations in the intensity of the note are called beats, and have their origin in interference. Should the forks not be vibrating with equal amplitudes, these effects would be less marked. As a beat occurs every time one fork gains a complete vibration over the other, the number of beats per second equals the difference in the frequencies of the forks. If *a* and *b*, Fig. 118, represent wave trains from independent sources, it is evident that 15 waves of a are emitted in the same time as 12 of b. From what has preceded, the simultaneous emission of both trains should result in three beats during this same time interval. A combination of these two wave trains by adding their ordinates does, in fact, show three beats.

182. Intensity Relations.—If the forks are vibrating with equal amplitudes, the intensity at the maximum of a beat is four times what it would be if either fork alone were sounding, since intensity varies as the square of the amplitude, and at the maximum the amplitude is doubled. It will be noticed that the average intensity and, therefore, the average energy sent out during the times of maximum and minimum is $\frac{4 + O}{2} = 2$, that is, twice the intensity due to a single fork, which is natural as two forks are vibrating. It can be shown that the mean intensity over a complete beat is twice the intensity due to a single fork. In other words, interference does not operate to create or destroy energy, but causes a redistribution of it.

Interference effects are possible wherever energy is propagated by waves. The effect of beats just described is only one kind of interference. There are numerous other types of interference, many of which can be studied more advantageously in the subject of Light. These effects are interesting not simply because they are in themselves a very positive proof that wave motions are involved, for no other manner of distributing energy permits interference effects, but because in numerous instances and in various fields interference effects have great practical value.

PROBLEMS

1. What is the velocity of sound in air at $-40°$?

2. Two tuning forks, sounding together, produce four beats per second. If one has a frequency of 480, the frequency of the other must be one of which two values? Can you think of a way of deciding which?

3. If simultaneous signals, by radio and underwater sound, are sent to sea from a lighthouse, how far distant is a vessel which receives these signals 5 seconds apart?

4. What is the reverberation period of a room 20 ft. x 19 ft. x 13 ft. when all the walls have an absorption coefficient of .025? What would the period become if the walls were covered with hair felt whose coefficient is .78?

5. A train, running at 50 miles per hour, approaches a crossing where a gong is ringing. If the gong's actual frequency is 420, what is its apparent frequency to a person on the train?

CHAPTER XII

VIBRATING SOURCES

183. The Vibrating Sources of Waves.—Before considering wave propagation further, it is desirable to consider more in detail the vibrating sources of the waves. This is because the vibrations of the particles composing the transmitting medium are copies of the vibrations of the source, and because the character of these vibrations determines the character of the waves. A study of vibrating sources will show that the vibrations are themselves frequently the result of wave propagation in the vibrating bodies or systems, and that the particular forms of vibration observed result from the propagation of one or more disturbances in the body or system. That several waves can be so propagated without permanently reacting upon one another in any way is a matter of common observation. Two or more pebbles dropped simultaneously into a quiet pool produce independent wave systems which, as they spread out, cross each other without being altered in any respect. Similarly, each musical instrument in an orchestra sends to the ear wave trains which are not altered in any way by the presence of the trains due to the notes of the other instruments, though all these waves pass through the same medium. Before discussing the vibration of a string from this point of view, it will be advantageous to consider the way in which an impulse may be transmitted along a string.

184. Propagation of a Transverse Disturbance Along a String.—In Fig. 119, let the lines *BC* represent a flexible string or rope fastened to solid supports at the ends. When a sharp blow upward is struck at *A*, the midpoint of the string, this point moves upward in the direction of the applied force and an elevation is produced as shown. Energy has been communicated to the string and its equilibrium has been disturbed. Experiment shows that there are sharp bends in the cord at the points s and u which travel outward toward the fixed ends with the same constant velocity. This is indicated by the two short horizontal arrows. The short vertical arrows indicate the directions of motion of the parts of the string at the bends, u and s.

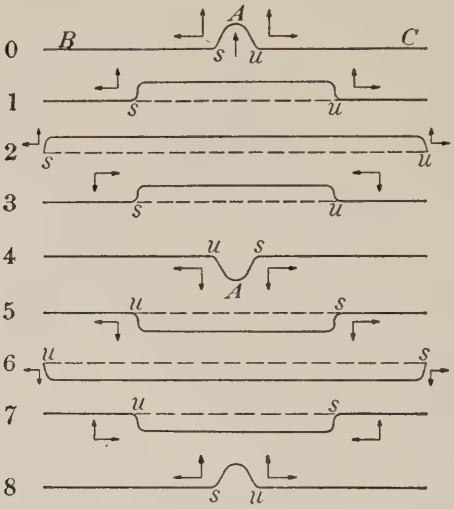


FIG. 119

The general idea back of the propagation of this disturbance along the string is as follows: The string being flexible—that is, without rigidity—the only force involved is the force F tending to stretch the string.

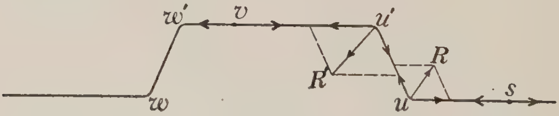


FIG. 120

At a point s, Fig. 120, the particle is pulled equally in both directions by forces each equal to F. The particle is accordingly in equilibrium. At the bend u, the particle is also under the action of two forces each equal to F, but they are no longer oppositely directed. They have, on the contrary, a resultant R which, acting on the particle at u, displaces it in the direction of R, thus moving the bend along to the next particle, etc. The particle u', which has been given an upward motion earlier in response to a similar force, finds itself acted upon by the two forces F, whose resultant is R', a force sufficient to stop the motion in the upward direction. Particles in the flat portion of the elevated part, as at v, are acted upon by equal and oppositely directed forces, as was the particle s. They are accordingly in equilibrium. This general explanation of the propagation of the bends u and u' toward the right is applicable to the bends w and w' which are propagated toward the left with the same velocity.

Upon arrival at a fixed end, the disturbance expends its energy in an elastic distortion of the string and support. Thus the energy becomes temporarily potential energy of distortion. The forces of restitution cause a recovery of the distorted portions of the system, with the result that the end of the string experiences a downward thrust which would cause a depression equal in amplitude to the incident elevation, were there no losses due to friction and other causes attending the reflection.

After reflection at C and B, as shown in line 2 of Fig. 119, the disturbances, now depressions, simply bring the elevated string back to its original position of equilibrium by the time they meet at A, line 4. Then, passing each other, each depression, now running along the string in its original position, produces a downward displacement. When u and s reach the ends, as in line 6, a process just the reverse of that in line 3 occurs, and the reflected elevations are seen in line 7, bringing the string back to its normal position.

At A these two reflected elevations meet, and A is urged upward, line 8, in a way which produces an effect quite similar to that of the original upward blow, as shown in line 0. Since now the situation in line 8 is identical with that in line 0, the entire succession of configurations of the string shown in Fig.

119 will continue to repeat itself in the same time as before. This means that the string is a *vibrating body.*

185. Natural Periods of Vibration.—The time required to run through this cycle of transformations, Fig. 119, is a natural period of vibration of the string, since it is determined wholly by the properties of the string itself. It is noticed that the disturbance u travels twice the length of the string in this cycle. If L represents the length of the string and V the velocity of propagation of the disturbance u, then the period of vibration is

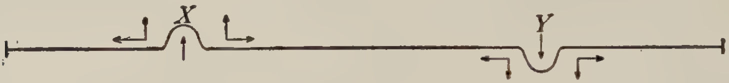


FIG. 121

$$T_1 = \frac{2L}{V}.$$

Should the string be struck simultaneously at the points X and Y midway between the center and ends of the string, upward at X and downward at Y, as in Fig. 121, there will be a double set of disturbances propagated, each similar to the single set in Fig. 119. It will be found that the period required for this configuration to repeat itself is just one-half that required by that of Fig. 119. That is,

$$T_2 = \frac{1}{2} T_1 = \frac{1}{2} \cdot \frac{2\,L}{V}.$$

Similarly, if the string is struck simultaneously at the points X, Y and Z, Fig. 122, in the directions indicated, a cycle of displacements will result in a time only one-third as long as that required by the cycle of Fig. 119. Hence, in this case,

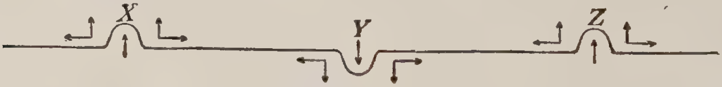


FIG. 122

$$T_3 = \frac{1}{3} T_1 = \frac{1}{3} \cdot \frac{2\,L}{V}.$$

These periods are all *natural periods* of the string. The following equations for these periods written in succession suggest others of the same type:

$$T_1 = \frac{2L}{V}$$

$$T_2 = \frac{1}{2} \cdot \frac{2L}{V}$$

$$T_3 = \frac{1}{3} \cdot \frac{2L}{V}$$

$$T_4 = \frac{1}{4} \cdot \frac{2L}{V}, \text{ etc.}$$

Experiment shows that strings do have natural vibrations whose periods are those given by the 3rd and 4th and following equations. These equations may be summarized by the single general equation,

$$T = \frac{1}{K} \cdot \frac{2L}{V}, \text{ or } KT = \frac{2L}{V},$$

where K takes in succession the values of 1, 2, 3, 4, etc. This equation represents therefore all of the possible *natural periods* of vibration which a flexible cord, fastened at both ends to solid supports, may possess. The longest period, T_1, is the period of the *fundamental* vibration; the other periods T_2, T_3, etc., are the periods of the *overtones*. When the period of the fundamental, as in this case, is an *integral* multiple of the periods of the overtones, the latter are also called *harmonics*.

Problem 1.—Instead of providing the elevation at the midpoint A, as in line 0, Fig. 119, produce it at the quarter position halfway between the end B and the midpoint A. Trace out the propagation of the disturbance in a similar way, and show that in this case also the string has the same natural period of vibration as before.

Problem 2.—Repeat the above, but start the disturbance by a blow close to the end B.

186. Resonance.—Consider Fig. 119. The series of configurations shown in lines 0 to 8 has in line 8 returned to the original configuration of line 0. The string will again begin to pass through the same succession of shapes. If, at the instant it has the form of 0 or 8, it is again struck a blow identical in all respects to the original blow of line 0, work will again be done on the system and the system will store up a greater energy of vibration. This will show itself by having the displacements at s and u greater in amplitude. If this blow is struck periodically whenever the system returns to the configuration of line 0, increased energy will be stored in the vibrating string and the elevations at s and u will increase with each blow. However, a succession of impulses, having the period T, of the string will not store up an unlimited amount of energy. Frictional and other effects cause losses of energy both as the distortions travel along the string and as they are reflected at the ends. A state of equilibrium is reached when these losses, due to the increased amplitudes of vibration, have grown in magnitude until they occur at a rate equal to that at which energy is given the string through work done on it by the periodic impulses.

When a body or a system has its energy of vibration increased by blows or impulses whose periodicity is that of any of the natural periods of vibration of the system, a *state of resonance* is said to exist between the periodic force and the system, and the system vibrates in the mode characteristic of this frequency.

187. Other Vibrating Systems.—Most objects and mechanical systems, as well as many electric circuits, have these natural periods of vibration. Thus bridges, the rotating shafts and bodies of automobiles, the diaphragms of telephones, the horns of phonographs, the atoms of molecules, bells, steel girders, the circuits commonly used in radio transmission, all possess natural periods of vibration and can be thrown into vigorous states of vibration through resonance. Many of these systems are studied for the purpose of suppressing their vibrations; others, for the purpose of developing their vibrational properties. A study of the vibrations of atoms within molecules leads to an understanding of atomic and molecular structures. Vibration and resonance are phenomena of the greatest theoretical and practical importance. Clear conceptions of the fundamental ideas involved in these processes are necessary before studies can be made of such systems as are enumerated above. These fundamental ideas are best obtained from simple systems. It is because vibrating strings demonstrate these ideas so clearly and directly that they are being so fully studied.

188. Standing Waves.—When a string is made to vibrate by a single blow, as shown in Fig. 119, the characteristic modes of vibration soon disappear, due to the distribution of the energy among the various possible natural modes of vibration. When, however, periodic impulses, equal in their frequency to any of the natural frequencies of the string, are applied to the string, as discussed under Resonance, the mode of vibration characteristic of this frequency is so much intensified through resonance that it alone appears to be present. Vibrating strings, set in vigorous vibration through resonance, are thus advantageously studied.

To produce the vigorous vibrations through resonance, it is sufficient to apply a periodic force of the requisite frequency anywhere along the string. The energy, communicated to one portion of the string, distributes itself by propagation throughout the string. Practically, the most convenient method is to tie one end of the string to a prong of a tuning fork which is mounted parallel to the length of the string and which vibrates transversely with respect to the string, as illustrated by Fig. 123. Though the end of the string is necessarily a place of small amplitude of vibration, nevertheless, as the experiment shows, energy is rapidly conveyed from the fork to the string when the frequency of vibration of the fork is any one of the natural frequencies of the string.

The impulses given to the string by the vibrating fork differ in form from those described in §183, but are propagated in the same way. They will repeat the original configuration in periods equal to those given in §184. Thus, Fig. 123 shows a series of positions of the string when the impulses repeat themselves in each fundamental period of the string. This time is given by the formula

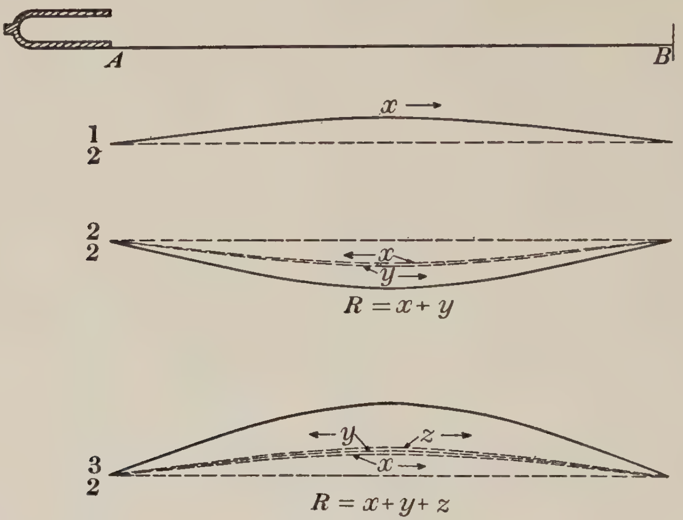


Fig. 123

$$T = \frac{2L}{V}.$$

The successive impulses enter the string at one end and return from the other end after reflection with change of phase. The resultant motion may be found by adding the displacements of these wave trains which are traveling in opposite directions. Fig. 123 shows the successive components and their sum for different instants.

Fig. 124

If the rate of vibration of the string is sufficiently rapid, it may make a complete vibration before the residual effect on the eye disappears, with the result that the string presents the appearance of a solid spindle, as illustrated in Fig. 124.

The maximum amplitude of this spindle is determined by the condition for equilibrium. This condition is reached when the string receives energy from the fork at the same rate as it loses energy as a result of friction and radiation. While this analysis of the phenomena is based upon the propagation of disturbances, both elevations and depressions, along the string, the eye simply sees the string rise and fall. Propagation not being evident, the waves are said to be stationary. Figs. 125 and 126 give a similar description of the vibration when the impulse has the frequency of the next shorter natural period of the string, that is, twice the frequency of the fundamental. The third natural frequency of the string has a period $T_3 = \frac{1}{3} \cdot \frac{2L}{V}$.

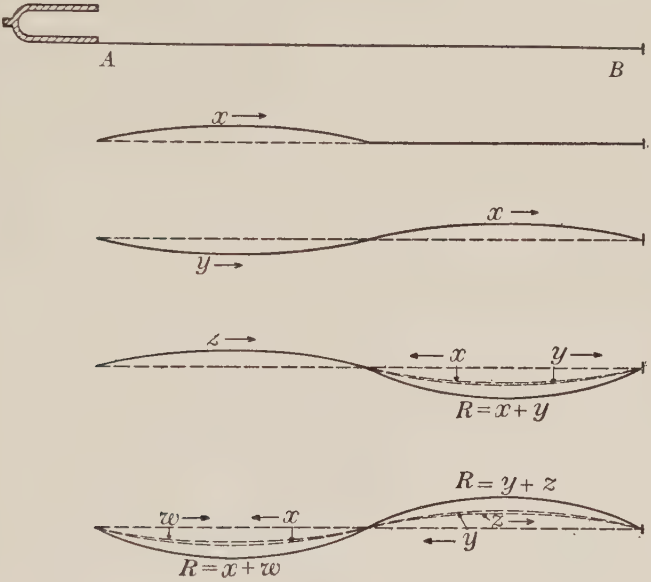


FIG. 125

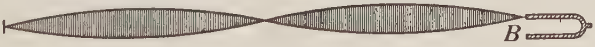


FIG. 126

If a fork of this frequency is used, the time of one vibration will carry the disturbance two-thirds of the way to B, $\left(T_3 = \frac{2}{3} \cdot \frac{L}{V}\right)$ and a development similar to the preceding will show that the string will vibrate in three spindles, as illustrated in Fig. 127.

189. Nodes and Antinodes.—If one examines the figures which have just been described, it will be noticed that the middle point of the spindles is a point in the string where the two wave trains always pass in identical phase. The maximum displacement at this point is the sum of the maximum displacements of the two wave trains. This point is called an *antinode.* The point between the spindles is a point at which the waves always pass with a difference of phase of 180°, and if their amplitudes are alike, the sum of their displacements will always be zero. Such a point is called a *node.* The distance from node to node or from antinode to antinode is a half wave length.

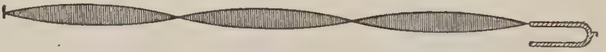


Fig. 127

It will be noted that all parts of the string in a single spindle pass through zero and rise to a maximum together. Their resultant motions are in phase but have different amplitudes. Successive spindles differ in phase by 180°. One rises as the other falls.

The states of resonance so far discussed may be summarized as follows: When the string vibrates in a single spindle, the fork makes one vibration while the wave runs down the string and back, $T_1 = \frac{2L}{V}$, K being 1. When the string vibrates in two spindles, the fork makes two vibrations while the wave runs down and back, $2T_2 = \frac{2L}{V}$, $K = 2$. In general, the number of spindles equals the number of vibrations that the fork makes while the wave runs down and back over the string.

190. Other Ways of Obtaining Resonance Conditions.—So far resonance has been produced by making the period of the applied impulse equal to any one of the natural periods of the string. It is possible to have a succession of states of resonance between a string and a fork with but a single period of vibration. With one end of the string tied to the end of the prong as before, the length of the string is regularly increased by taking it in the fingers close to the fork and gradually moving away. At a certain length the string vibrates vigorously in a single spindle. This means that $T = \frac{2L_1}{V}$, or the fork makes one vibration while the wave runs to the fingers and back. Continuing to increase the length of the string as before, when a length equal to double the first is reached, the string vibrates vigorously in two spindles. This means $2T = \frac{2L_2}{V}$, or the fork makes two vibrations while the wave runs down and back. Thus by increasing the length of the string to 3, 4, 5, etc., times its first length, resonance is obtained, the fork making 3, 4, 5, etc., vibrations in the time the wave takes to go down the string and back.

When the period T of the fork was varied to produce resonance, the length L of the string and the velocity V with which the wave traveled along the string, remained constant. In the second case of resonance, T and V were constant and L varied. It is quite possible to produce the various conditions of resonance by varying V while T and L are constant.

The velocity $V = \sqrt{\frac{F}{m}}$, where F is the force on the string and m the mass per unit length of the string. Thus with a given string, where m is constant, V can be varied by changing F. The string tied to the fork may be stretched accordingly till it shows a state of resonance by vibrating in a single spindle. This means the fork vibrates once while the wave goes down and back. By decreasing the tension sufficiently, the velocity may be made half of the previous velocity. This will make the time for a wave to run down and back over the string twice as long, and 2 vibrations of the fork will be made, which means the string will vibrate in 2 spindles. Similarly, if the tension is decreased by steps till the velocity becomes 1/3, 1/4, 1/5, etc., its first value, the fork will make 3, 4, 5, etc., vibrations during the time required by the wave to run down and back, or the string will vibrate with 3, 4, 5, etc., spindles.

More generally, all these quantities T, L and V may be adjusted to cause a state of resonance to exist between the fork and string.

191. The Laws of Vibrating Strings.—If the value of V in the equation, $V = \sqrt{\frac{F}{m}}$, is substituted in the general equation of resonance, $KT = \frac{2L}{V}$, the resulting equation contains all of the quantities which determine the period of vibration of a string.

$$KT = \frac{2L}{\sqrt{\frac{F}{m}}};$$

$$T = \frac{2L}{K\sqrt{\frac{F}{m}}};$$

$$n = \frac{1}{T} = \frac{K}{2L}\sqrt{\frac{F}{m}}.$$

When the string vibrates in one spindle, $K = 1$, or

$$n = \frac{1}{2L}\sqrt{\frac{F}{m}}.$$

In this case it is seen that the frequency n is inversely proportional to the length of the string L. Thus musical instruments intended for low notes have long strings. The double bass is an example. n varies directly as the square root of the stretching force F. Thus quadrupling the tension doubles the frequency

n. Strings giving out notes of very high frequency would be under great tension if it were not possible to make them quite short.

n varies inversely as the square root of the mass per unit length. The frequency can be cut down by increasing this mass. This is done in the case of the bass strings on many stringed instruments by winding them with heavy wire.

192. Overtones. Quality.—The configurations of a string as it vibrates in its natural periods have thus been determined. When strings of stringed instruments are thrown into vibration they invariably vibrate in all of these characteristic ways at the same time. The fundamental frequency, in which the vibrations are the most intense, is the lowest, and is produced when the string vibrates in a single spindle. The notes of higher frequency, due to the string vibrating in 2, 3 and more spindles, are called overtones and have frequencies 2, 3, 4, etc., times that of the fundamental. When a string vibrates in all these modes simultaneously, its resultant vibration is very complex, and the sound wave which it sends out is complex. Usually the fundamental is more intense than any of the overtones and determines the pitch of the note. The superposition of the many overtones upon the fundamental very materially modifies the character of the note and gives it its *quality*.

193. Other Vibrating Objects.—A single blow, or two or more blows struck simultaneously, will set other bodies or systems vibrating with frequencies proper to them. A single blow upon a hollow tube will cause the enclosed air to vibrate with its proper frequencies, one of these often being sufficiently intense to be distinguished by the ear as a note of definite pitch. Anvils ring with the blows of the hammer, and metal plates and bells vibrate when struck. In these latter instances the resultant vibrations, again combinations of the possible vibrations proper to the objects, are very complicated. The higher frequencies are not in general integral multiples of the lowest or fundamental frequency, as was the case with strings.

194. The Natural Vibrations of Air Columns.—If an open tube is struck a sharp blow on one end with the flat of the hand, the air contained in the tube gives out a sound of definite pitch. In accordance with the knowledge gained from studying strings, this would be interpreted as showing that the enclosed air column has a natural frequency of vibration. It is, in fact, quite possible to discover the natural frequencies for such air columns in a manner analogous to that used in the case of the string. In a gas, such as air, the disturbances which are propagated are condensations and rarefactions, and the following of individual disturbances becomes confusing. For this reason the natural periods of vibration of air columns will be developed by the aid of the principle of resonance. This is possible since, as in the case of the strings, the various frequencies of the periodic impulse causing resonance are the same here as the natural frequencies of vibration of the air columns. Here also the cycles of successive states which constitute the various types of vibration natural to air columns have their cause in the wave disturbances which are propagated back and forth through the air column and which through interference cause the formation of stationary waves with nodes and antinodes. Before entering upon a detailed consideration of this problem, we must first consider the reflection of compressional waves which consist of condensations and rarefactions.

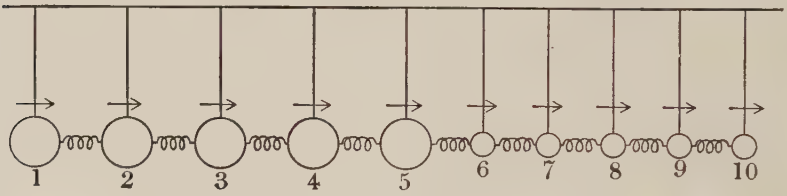


FIG. 128

195. Reflection of Compressional Waves.—A series of balls are hung by strings, as shown in Fig. 128. Half of them are heavy and half are light. The balls are connected by similar springs as shown. When ball 1 is given a displacement to the right, represented by the arrow above it, the spring between it and 2 is compressed more than the spring between 2 and 3. This difference in compression causes ball 2 to move to the right, and so on till ball 5 is reached. As 5 moves to the right, the smaller inertia of ball 6 offers less resistance to the movement of 5 than the preceding balls have experienced. Therefore 5 moves farther to the right than any of the preceding balls. Thus a condensation has been propagated from 1 to 5. This larger displacement of 5 in the direction of 6 means that the spring between 5 and 4 has been elongated. As a result, 4 moves toward 5, causing in turn the elongation of the spring between 4 and 3, etc. An elongation or rarefaction thus runs back from 5 toward 1. Part of the energy which was carried from 1 to 5 by the propagation of the condensation has been returned as a rarefaction. The medium of propagation from 6 on is a lighter medium. A condensation incident upon a lighter medium is in part reflected as a rarefaction and in part continues on in the lighter medium as a condensation.

Displacing 10 toward 9 by a small amount results in a compression of the spring between 10 and 9, and this compression is transmitted as before till the disturbance reaches 6. 5 is not displaced forward the same amount as the preceding balls because its larger mass gives it a greater inertia. The result is that the spring between 5 and 6 becomes more compressed than the spring between 6 and 7 and causes 6 to move toward 7. A condensation thus runs back from 5 to 10. A condensation coming to a heavier medium is in part reflected as a condensation and in part continues as a condensation in the heavier medium.

The chief point of the preceding is that a disturbance propagated along a uniform medium suffers partial reflection of its energy when there is a change of medium, and that a condensation is reflected as a rarefaction or as a condensation depending upon whether the reflection occurs against a lighter or a heavier medium.

It can be shown that a rarefaction (a greater elongation of the springs) will be propagated in a similar way, and that it will also suffer partial reflection upon coming to a change in the medium of propagation. A rarefaction is partially reflected as a condensation when the reflection is against a lighter medium, and as a rarefaction when against a denser medium. The student should convince himself of the truth of these statements, using Fig. 128 for this purpose.

196. Vibrating Air Columns.—Pipes open at both ends. If a vibrating reed of adjustable period moves from *a* toward *b*, as in Fig. 129, it produces a region of condensation on its forward side. Such part of the condensation as enters the pipe travels along the pipe much as the condensations of the springs were transmitted, the volume elasticity of the air corresponding to the elasticity of the springs. When such a region of condensation reaches the open end of the pipe, the opportunity it has of expanding in all directions causes a partial reflection as if against a lighter medium. A part of the energy is therefore reflected as a rarefaction. This in its turn, reaching the open end next the reed, is partially reflected as a condensation. If there is to be a state of resonance between the vibrating reed and the air column, this condensation starting down the pipe a second time must be added to by the reed. The reed must be producing a condensation, that is, moving from *a* toward *b*. When last considered, the reed was moving from *a* toward *b*. If now it also moves in this same direction, it must have gone at least from *a* to *b* and back from *b* to *a* while the disturbance ran down and back through the pipe. That is, the time of a single vibration of the reed is equal to the time required for the disturbance to run down and back through the pipe, or $T = \frac{2L}{V}$, where L is the length of the pipe and V is the velocity with which the disturbance travels in the pipe.

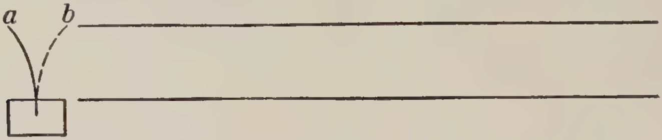


Fig. 129

A second supposition which could be made is that, instead of making a single vibration, the reed has made two vibrations. This will again bring it to the position *a*, ready to move toward *b*. This second condition of resonance requires, therefore, the reed to make twice as many vibrations per second as in the first case, or $2T_2 = \frac{2L}{V}\cdot$ Its period is thus one-half its former period. $T_2 = T/2$, since now 2 vibrations are made while the disturbance goes down and back through the pipe. A reed making three vibrations in this time will also produce the required additive effect, and $3T_3 = \frac{2L}{V}\cdot$ In general, if the period of the reed is adjusted so that it can complete *any* integral number of vibrations while the disturbance runs down and back through the pipe, there will be resonance. This relation is expressed by the equation

$$KT = \frac{2L}{V},$$

where K may have the values of 1, 2, 3, 4, 5, etc. This same equation was obtained for the vibrating string.

197. The Stationary Waves of Open Pipes.—When $K = 1$, that is, when resonance results because the reed makes one complete vibration in the time $\frac{2L}{V}$, the distance traveled, $2L$, in the time of this single vibration must be a wave length of the wave disturbance being propagated in the tube. L is therefore half the wave length. By the time the reed has finished its first half vibration from *a* to *b*, the condensation will have reached the far end of the pipe one-half wave length distant from the reed. As the reed makes its return half vibration from *b* to *a*, it is sending a rarefaction into the near end of the tube, while the original condensation is being partly reflected as a rarefaction at the far end of the tube. The rarefaction which approaches the center from the right travels toward the center by having the particles move in their vibrations toward the right. Similarly, the rarefaction approaching the center from the left does so by having the air particles move in their vibrations to the left. Thus, as the two rarefactions approach the center, they form a double rarefaction. Moreover, this layer, urged equally to vibrate to the right and to the left, remains at rest.

These two rarefactions will pass each other and be partially reflected again at the open ends as condensations, the one at the reed end being reinforced. To propagate these condensations toward the midpoint, the particles in their vibratory paths must be moving in the same direction as the condensations. Thus, as the midpoint is approached, the particles of the air move toward this point in their vibrations, producing a "double" condensation or a place of maximum density. As before, the air layer at the middle urged to move equally both ways remains at rest; thus the middle is a place of minimum vibration of the air layers. It is therefore a place where the waves traveling in opposite directions urge the particles to vibrate in opposite phase. Such places are nodes. Thus on opposite sides of a node the particles move in opposite directions in their vibratory motions. In the string there was a similar situation; on one side the string moved up, on the other down. As already pointed out the center, now shown to be a node, is successively a place of minimum and maximum density and therefore a place of maximum change in density. Nodes in air columns are regions of minimum vibration and maximum change of density.

As there is a node at the center of the air column and the entire column is one-half wave length long, the two ends must be antinodes. These are places of maximum amplitude of vibration of the air particles. This is in accordance with the conditions of the experiment. The antinodes are places of minimum changes of density, as would be expected, as the pressure at the open ends cannot vary greatly from atmospheric.

For $K = 1$ the condition of resonance is represented graphically by Fig. 130. The lengths of the arrows represent the amplitudes of vibrations of the particles, being very small in the neighborhood of the node N, and large at the antinodes, V. All particles between the node and either antinode vibrate in phase, all toward the node transmitting a condensation to the node, all away from it transmitting a rarefaction to the node.

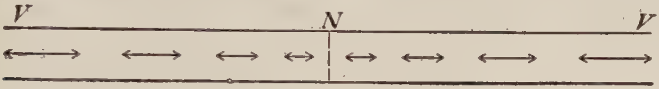


Fig. 130

When $K = 2$ the stationary wave is represented graphically in Fig. 131, where the arrows represent displacements at an instant when they have maximum values. N_1 is a region of maximum density and N_2 is a region of minimum density. A half period later the arrows are reversed, and N_1 becomes a region of minimum density. The student should convince himself of this by developing the method used when $K = 1$ to meet this case. Also represent by similar figures the cases $K = 3$ and $K = 4$.

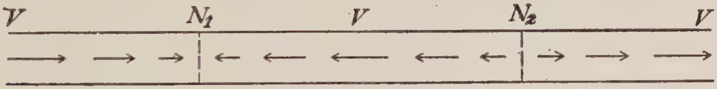


Fig. 131

198. The Overtones Characteristic of Air Columns in Pipes With Open Ends.—From the preceding development it is evident that air columns of the above kind have many natural frequencies of vibration. Since the periods of these are given by the equation $T = \frac{1}{K} \cdot \frac{2L}{V}$, where $K = 1, 2, 3,$ etc., the frequency n will vary as the successive integers, 1, 2, 3, 4, etc. Thus $n = \frac{1}{T} = K \cdot \frac{V}{2L}$. This means that pipes with open ends may produce the complete series of harmonics.

199. The Natural Periods of Vibration of Air Columns in Pipes Closed at One End.—An air column of this kind is shown in Fig. 132, where the pipe is open at the reed end and closed at the other. When the reed moves from *a* toward *b* the condensation formed in front of the moving reed enters the tube and, traveling its length, is reflected by the wall closing the end. The circumstances of this reflection are similar to those of the springs where the reflections occurred against a heavier medium. The condensation accordingly starts back after reflection as a condensation. Reaching the end near the reed, it is partially reflected as a rarefaction. To amplify this, the reed has to be moving from *b* toward *a*. As, when last considered, it was moving from *a* toward *b*, the simplest supposition regarding its motion while the disturbance has run down the tube and back is that it has gone from *a* to *b*. That is, it has made one-half a vibration. Thus

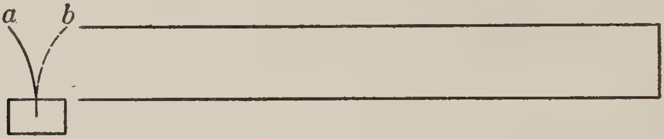


FIG. 132

$$\frac{1}{2}T_1 = \frac{2L}{V}.$$

If the reed should make 3 half vibrations, it would again be at *b* ready to move toward *a* as the rarefaction starts down the pipe. Thus

$$\frac{3}{2}T_2 = \frac{2L}{V}.$$

It will be evident upon consideration that any odd number of half vibrations will bring the reed to the position *b* in the time $\frac{2L}{V}$. Now $(2n - 1)$ is an expression for the successive odd numbers 1, 3, 5, 7, 9, etc., when n has the values 1, 2, 3, 4, etc.; therefore

$$\frac{1}{2}T_1 = \frac{2L}{V},$$

$$\frac{3}{2}T_2 = \frac{2L}{V},$$

$$\frac{5}{2}T_3 = \frac{2L}{V},$$

and other similar equations may be expressed generally as

$$\frac{(2n-1)}{2}T = \frac{2L}{V}.$$

Since $\frac{2L}{V}$ is the same in all cases, $T_2 = \frac{T_1}{3}$ and $T_3 = \frac{T_1}{5}$, or $n_2 = 3n_1$ and $n_3 = 5n_1$, or the periods of the natural vibrations of such air columns vary as 1, 1/3, 1/5, etc., while the frequencies vary as 1, 3, 5, 7, etc.

200. Standing Waves in Air Columns Contained in Pipes Closed at One End.—As before, and for the same reasons, the open end of the pipe near the reed is an antinode. It is evident that the closed end of the pipe must be a place of minimum vibration since particles in contact with the end cannot vibrate. These conclusions are in agreement with the conditions of resonance. When $\frac{1}{2}T_1 = \frac{2L}{V}$, the wave length is $4L$, since $2L$ is the distance traveled during a half vibration. One-quarter wave length is then equal to the length of the pipe, that is, from an antinode to the next node. Fig. 133 is a diagram of the stationary wave for this case of resonance, when the amplitudes of the longitudinal vibrations of the air particles are represented by the double arrows. When the vibratory motions are all toward the node, a region of maximum density is form-

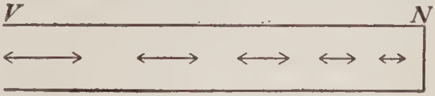


FIG. 133

ing. When the motions are all away from the node, a rarefaction is forming.

When the reed makes 3/2 vibrations instead of 1/2, the wave length will be only 1/3 as long. Therefore in this case of resonance, instead of going the length of the pipe to pass from antinode V_1 to the next node N_1, it will be necessary to go but 1/3 this distance, as shown in Fig. 134. Here the wave is shown when there is a region of maximum density forming at the first node N_1 and a region of minimum density forming at N_2. It will be noticed that the vibrations are oppositely directed on the two sides of N_1, and that in the entire segment from N_1 to N_2 all vibrations have the same direction and are in the same phase.

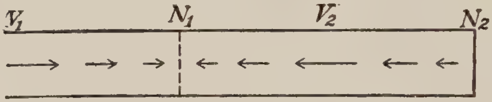


Fig. 134

When the reed makes 5/2 vibrations in place of 1/2 for resonance, the wave lengths of the resulting standing wave are only 1/5 as long, and the distance from V_1 to N_1 is 1/5 that of the first case. Construct the figure to represent this case.

201. The Importance of Resonance.—The phenomenon of resonance, while it occurs in many instances in nature, is intentionally brought about in many practical operations, because it is a device by which energy can be rapidly withdrawn from a vibrating source and stored in the system which is in resonance with it. If energy is withdrawn more rapidly from the vibrating source when connected to the system in resonance than when not so connected, it must be that more work is done in a given time when connected than when unconnected. That this is a fact, the student should convince himself by seeing that when there is resonance all the displacements of the vibrating source are made in opposition to an extra opposing force arising in the system put into vibration through resonance. As an illustration, when a trough is being reflected as a crest at the vibrating fork, the string exerts a greater downward force on the vibrator which is moving upward than would be the case if there were no such reflection. If the fork is electrically driven it maintains a constant amplitude of vibration. If there is resonance, every reflection at the fork is of such a nature that the forces against which the fork produces these constant displacements are larger in general than would be the case if there were no resonance. The fork accordingly does more work in a given time when there is resonance. The student should convince himself of the truth of this for each kind of resonance discussed.

202. Quality.—This property of a note already briefly treated has been ascribed to the complexity of the sound wave. Having discussed at length the natural frequencies of vibration and the forms of vibration of strings and air columns, it is now possible to be more definite about this characteristic property of musical notes.

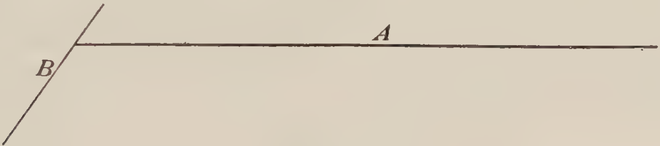


Fig. 135

A string fastened at both ends vibrates simultaneously in all of its many natural frequencies, every one of which is an integral multiple of its lowest or fundamental frequency. This very complex vibration results in the formation of a correspondingly complex wave. It is easier to visualize such a wave if its form can be constructed. The following method, illustrated by Fig. 135, permits this. If waves are sent along a string *A*, so long that no reflected waves need be considered, by using as a source of vibration a string *B* vibrating in the manner already discussed, a transverse wave will run along the string *A*, whose complexity corresponds to the complexity of the vibration of the string *B*, where there are standing waves of all permissible frequencies. If each of these standing waves is considered as sending along *A* a simple wave characteristic of its own frequency, there will be on A a number of single waves whose resultant can be readily formed and which will be the same as the complex wave really sent out by B.

For simplicity, consider B to vibrate with but two natural frequencies, the fundamental and the first overtone. Then there will be the simple waves on A due to these vibrations. The wave due to the fundamental is numbered 1, Fig. 136. The wave due to the first overtone is numbered 2, and is one-half as long as the fundamental wave, since the period is one-half as great. The resultant, R, of these simple waves is found by adding, with due regard to sign, the displacements due to the individual waves.

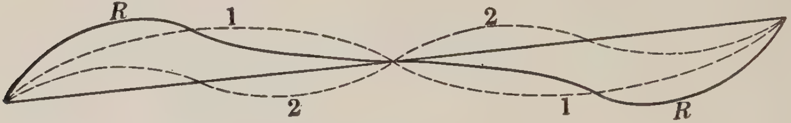


FIG. 136

It will be evident that the character of this resultant wave must change with the number and intensity of the overtones. Thus, the quality of the notes from organ pipes open at both ends will differ from the quality of notes from organ pipes open at one end only. In the case of the open pipes the air columns have all the overtones whose frequencies are integral multiples of that of the fundamental, while the closed pipes have only those overtones whose frequencies are the integral odd multiples of the fundamental. Voices differ in quality because, among other reasons, the air chambers of the chest, throat, and mouth differ in individuals. This difference in resonating air chambers produces a difference in the relative intensities of the overtones and possibly also a difference in the number of the overtones present. This composite note is generally much more pleasing than the fundamental would be, though certain of the overtones cause discords which are suppressed when possible.

With strings and air columns the frequencies of the overtones are integral multiples of the fundamental frequency. Such overtones are called harmonics. With bells and vibrating plates the frequencies of the overtones are not generally integral multiples of those of the fundamental, and they are not called harmonics.

PROBLEMS

1. A string under a stretching force of 16 lbs. vibrates 64 times per second. What force must be used to cause it to vibrate 86 times per second?

2. The tuning fork, Fig. 123, makes 20 vibrations per second. The string is 3 meters long; what is the velocity of the wave in the cord?

3. If the string in the preceding problem has a mass of 1/10 gm per cm length, what will be the stretching force?

4. With the temperature at 16° C., what is the length of an open organ pipe whose frequency is 280 vibrations per sec.?

5. How much would the frequency of this pipe change if the temperature fell to freezing, 0°C.?

6. What are the frequencies of the fundamental and the first two overtones of a closed pipe 50 cms long when the temperature is 20° C.?

CHAPTER XIII

PHYSICAL THEORY OF MUSIC

203. Musical Scales.—The pitch of a musical sound depends upon its vibration frequency, being higher as the frequency is higher. The significance of certain familiar musical relations may be studied by means of a siren, mentioned in §169, which is a circular metal disk through which are bored several concentric circular rows of equidistant holes. The plate is mounted on the shaft of a motor so that it may be caused to rotate rapidly. When air is blown through one of the rows of holes, a sound is produced whose frequency is the product of the number of holes on this circumference and the number of revolutions per second. If there were four such rows of holes having 24, 30, 36, and 48, respectively, as the number of holes per row, it is evident that the frequencies of the sounds produced by blowing air through the different rows would have the ratios 4 : 5 : 6 : 8. As we listen to these sounds we recognize them as having almost exactly the same relationship as the notes C, E, G, and c' as produced by the piano. The *musical interval* between two notes is measured by the ratio of their vibration frequencies; thus the interval between C and E is 5/4, and the interval which we know as the octave is the ratio 2/1.

Studies such as that mentioned with the siren lead to the conclusion that sounds having frequency ratios in which only small numbers occur produce a harmonious effect when sounded together. If, on the other hand, the ratio is not made up of small numbers, the combination is discordant. Two notes whose frequencies are in the ratio of 20: 21, when heard together, are distinctly disagreeable. This psychological effect is probably related in some way to the number of beats produced by the interference of the two sounds.

Three notes whose vibration frequencies are in the ratio 4 : 5 : 6 constitute a major triad. The diatonic scale is built upon three such triads. In the key of C, the three triads are

$$\boxed{\left.\begin{array}{l} C : E : G \\ G : B : D \\ F : A : c \end{array}\right\} = 4 : 5 : 6.}$$

If we assume some number, such as 72, to represent the frequency of the key note C, we may by means of these ratios compute the frequency of each of the other notes of the scale. We shall at the same time find the scale ratios for the different notes and the interval between each note and the one next above it in pitch.

Frequency	72	81	90	96	108	120	135	144
Name....	C	D	E	F	G	A	B	*c*
Syllable ..	Do	Re	Mi	Fa	Sol	La	Si	Do
Ratio....	$\frac{1}{1}$	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	$\frac{2}{1}$
Intervals .		$\frac{9}{8}$	$\frac{10}{9}$	$\frac{16}{15}$	$\frac{9}{8}$	$\frac{10}{9}$	$\frac{9}{8}$	$\frac{16}{15}$

It appears that there are three different intervals in the scale. The first two are called whole tones and the third is called a half tone. The intervals between the key note and each of the notes of the scale are named in accordance with the position of the upper note, *i.e.*, 5/4 is called a major third because E is the third note in the scale; 3/2 is called a fifth since G is the fifth note; and as *c*′ is the eighth note, the ratio 2/1 is called an octave. Other intervals are named in a similar manner.

204. The Minor Scale.—A minor triad consists of three notes whose frequencies bear the relation to each other of 10 : 12 : 15, and the minor scale is built up of three such triads. The vibration frequency of each note of the scale is found in a manner similar to that used with the major scale. The numbers and ratios are given in the following table, where 72 is taken as the frequency of the key note.

Frequency..	72		81		86.4		96		108		115.2		129.6		144
Name......	C		D		E		F		G		A		B		c
Ratio......	$\frac{1}{1}$		$\frac{9}{8}$		$\frac{6}{5}$		$\frac{4}{3}$		$\frac{3}{2}$		$\frac{8}{5}$		$\frac{9}{5}$		$\frac{2}{1}$
Interval....		$\frac{9}{8}$		$\frac{16}{15}$		$\frac{10}{9}$		$\frac{9}{8}$		$\frac{16}{15}$		$\frac{9}{8}$		$\frac{10}{9}$	

It will be observed that, to use the minor scale, three additional notes are needed that are not found in the major scale, *viz.*, the three below E, B, and A respectively of the major scale. Some new intervals also appear; the ratio 6/5 is called a minor third and 8/5 a minor sixth. The fifth is the same in both scales.

205. Transposition.—When some other note than C is taken as the key note, and the vibration ratios are applied to find the frequency of each note of the scale, it is found that some new notes must be added to the scale to supply all the necessary frequencies. If D were to be made the key note, G and B would be found correct, while A and E would differ but slightly from the required frequencies. However, F and C would be found to be too low, and it would be necessary to introduce two new notes, F sharp and C sharp, in order to produce music in the key of D.

206. The Tempered Scale.—Since each change of key requires the addition of new notes, both for the major and minor scales, it is evident that the number of notes for each octave becomes very large—so large, in fact, that in the case of instruments with fixed tones, such as pianos and organs, it becomes impossible to manipulate so many keys. For this reason, a compromise system, known as the system of equal temperament, has been adopted. In this system the whole tones are all made alike, and the ratio representing the half tone is the square root of that for the whole tone. Thus there are added five new notes to the octave, making thirteen in all. Referring to the table for the major scale, we see that the intervals between adjacent notes are all whole tones, except those between E and F and between B and c′ respectively. The black keys of the piano are inserted wherever these whole tone intervals occur, and the intervals between successive notes are all made equal. Suppose x to be this interval; then to find the frequency of C sharp we should multiply the frequency of C by x. Taking 72 as the frequency of C, and $72x$ the frequency of C sharp, $72x^2$ = the frequency of D, $72x^3$ = the frequency of E flat, $72x^4$ = the frequency of E, etc. Since there are twelve such intervals, it is evident that c′ has a frequency of $72x^{12}$. But C′ has a frequency of 144; therefore $x^{12} = 2$ and $x = \sqrt[12]{2}$. By the use of logarithms this ratio is readily found to be 1.059, *i.e.*, the frequency of any note produced by the piano is 1.059 times the frequency of the next lower note.

PROBLEMS

1. If middle C is assumed to consist of 256 vibrations per sec., (a) compute the vibration frequency of each of the other notes of the major triad; (b) compute the vibration frequency of each of the other notes of the minor triad; (c) find the interval between each pair of corresponding notes of the two triads.

2. Find the intervals between C and each of the notes of the major triad on the scale of equal temperament.

3. A string 100 cms long produces the note middle C. A bridge is placed under the string and adjusted till the string produces E of the same octave. Where is the bridge placed? Where should it be placed to produce the other note of the major triad?

4. If the stretching force of a certain string is 500 grams and it sounds middle C, what force will be necessary to tune it to D of the natural scale? To E of the piano scale, *i.e.*, the scale of equal temperament?

5. A guitar has a series of bridges so placed that their respective distances from the bridge at the other end of the strings are represented in inches by the following numbers:
E, 12.875; F, 12.1875; F♯, 11.5; G, 10.875; G♯, 10.25; A, 9.6875; A♯, 9.125; B, 8.5625; c, 8.0625; c♯, 7.625; d, 7.25; d♯, 6.8125; e, 6.4375.
What determines these distances? Make several computations to satisfy yourself that your explanation is correct.

6. Determine the length of an open organ pipe that is in unison with E above middle C of the piano when the temperature of the air is 25° C. End corrections are not to be considered in this problem.

HEAT

CHAPTER XIV

THERMOMETRY

207. Introduction.—Heat is a form of energy which is possessed by a body by virtue of the irregular motion of its molecules. We have already studied one type of molecular energy in the sound wave. The molecules of a medium through which a sound is passing are in a state of vibration. They are making excursions to and fro in the line of propagation of the wave. This sort of motion has a distinctly systematic character. It has everywhere a definite direction and at any point a definite amplitude and phase at a given time. The molecular motion we are now about to study has none of these characteristics. In a hot body the molecules are moving indiscriminately in all directions and with independent speeds. Collisions are frequent and result in continual changes in the velocities of the individual molecules. If the material is in a gaseous state, the molecules have considerable freedom of motion. An individual molecule may wander great distances in this random way. It is this motion which gives the gas the property of filling any container. In liquids the translation motion is somewhat more restricted, and only occasionally is the energy great enough to carry the molecule through the liquid surface. The molecules or atoms of a solid probably seldom leave their positions of equilibrium. They vibrate with characteristic frequencies but with no systematic phase relations. Such random motion cannot succeed in setting elastic bodies like the membrane of the ear in appreciable vibration. The impulses which it gives to other bodies are not properly timed to produce a cumulative vibratory effect.

They succeed, however, in passing on the thermal energy by setting the molecules of an adjacent body into similar irregular motion. Such transmission of energy is always taking place, for example, between a body and the surrounding air. In fact, it always takes place in both directions. Thermal equilibrium signifies only that the transfer in one direction has become equal to that in the other. In this case one commonly says that there is no flow of heat.

We may think of heat as a lower form of energy. Other more orderly types degenerate into it. This is the ultimate fate of sound vibrations. When a projectile loses its energy of translation at the target an equal amount of this chaotic form comes into being. The transformation of other forms of energy into heat takes place with such ease that we speak of it frequently as spontaneous. We have usually to exercise some ingenuity to prevent it. To reverse the transformation, that is, to produce order out of disorder, requires in general very special conditions.

It is not hard to see therefore that heat is an extremely common type of energy, and that its study will be of utmost importance for all science, particularly as almost every one of the properties of a body changes with its heat content.

208. Temperature.—In the experience of everyday life we form an idea of the meaning of the term, temperature. We possess a temperature sense by which we judge equality of temperature, and to some extent we are able to classify bodies as hot or cold. We are aware that bodies dissimilar either in mass or material may require very different amounts of heat energy to raise their temperature by equal amounts. For this reason, temperature cannot be used to indicate the amount of heat energy that has been given to a body except in certain special cases. It tells us rather whether heat will flow from one body to another when they are brought together. Two bodies have the same temperature if no heat flows from one to the other when they are brought closely in contact. If heat does flow from one to the other, we shall say that the body from which the heat flows had originally the higher temperature. Temperature is therefore analogous to water level. The height of the upper surface of water does not tell us how much water there is in the container, but it does indicate which way the water will flow if our container is connected to another.

From the point of view of the molecular theory, two bodies have the same temperature if, on collision of the molecules of one with those of the other, each body gains and loses equal amounts of energy.

209. Fixed Points.—In order that different observers may compare their measurements of temperature, it is necessary to agree on so-called fixed points. Many phenomena occur always at very definite temperatures when other conditions are duplicated. For example, water always freezes at the same temperature under given conditions of pressure. Its boiling point also is definite if the pressure is fixed. Moreover, these two changes of state take place well within the temperatures of ordinary experience and are comparatively easy to control experimentally. The boiling and freezing points of water under standard pressure have therefore been chosen as the fundamental fixed points, and their temperature difference is called the fundamental interval. All other temperatures are measured from one of these as a reference point.

210. Thermometers.—The fixed points furnish us with definite temperatures by means of which we may classify the temperatures of all other bodies as higher or lower than these or equal to them. To define a unit of temperature, however, requires a special instrument which is called a thermometer. Temperature is a quantity which, unlike length, cannot be measured by laying off a unit along any given interval. With our present knowledge we have no means of deciding that a body has a temperature midway between the temperatures of freezing and boiling water except by availing ourselves of some particular property of a particular material which changes with temperature, and by arbitrarily defining equal intervals of temperature as those which produce equal changes in this property. For example, we might decide that whenever a bar of iron increased in length by one per cent of its length in melting ice, its temperature had increased by a fixed number of units. It would simplify the problem greatly if all properties of all materials would yield the same results when used in this way as thermometers. Since this is not the case, we are forced to make an arbitrary choice. If we choose an exceptional property of some material, it is clear that the statement of any observations made, using it as a thermometer, will contain not only the complexities of the problem in hand but also the peculiarities of the thermometric substance. It is wise, therefore, to choose a thermometer out of as large a group as possible, the members of which yield the same results.

Later we shall become acquainted with an important and far-reaching principle called the second law of thermodynamics, which defines temperature in terms of a property which is common to all thermal processes. By means of it, all other temperature determinations may be restated and freed from the special complication of the thermometric substance. It does not provide us with a practical thermometer, but it gives us a means of deciding whether or not a thermometer which we find convenient departs widely from the "thermodynamic scale," and shows us how to correct for its deviations.

211. Mercury Thermometers.—For measurements of temperature in the ordinary range, the instrument in most frequent use is an expansion thermometer of mercury in a glass container. The glass container consists of a bulb of suitable size and, rising from this, a capillary tube of uniform bore. This is partly filled with mercury and sealed. The thermometer is placed successively in baths of saturated steam and melting ice, and the position to which the mercury rises in each case marked on the tube. The distance between these points is then divided into equal parts.

212. Determination of Fixed Points.—After the glass container has been filled with mercury, it is heated to a temperature somewhat higher than any at which the thermometer will subsequently be used, and sealed off. On cooling, the mercury recedes and leaves a space above it in the tube which is free from air.

When glass has been heated to temperatures sufficient for forming the bulb and stem it returns slowly to its normal dimensions. Different types of glass possess this "thermal after-effect" in varying amounts, but none are free from it. Before any determinations of the fixed points are made, the glass is allowed to recover for some time. A slight contraction continues, however, for many years, and the fixed points should be checked frequently and correction made for their displacement. This involves the same experimental procedure as that in their original determination.

To determine the boiling point, the thermometer is completely immersed in a bath of steam which in turn is surrounded by a steam jacket. Care is taken not to allow the bulb to dip into the boiling water which might easily have a higher temperature because of accidental impurities. The vapor, on the other hand, is also in contact with drops of condensed liquid on the walls of the dish and, as we shall learn later, this insures a definite temperature when the pressure is fixed. A manometer whose two sides are in contact with the vapor and the outside atmosphere records any difference in pressure. This difference is applied as a correction to the barometric reading, and the temperature is then determined from boiling point tables.

To determine the freezing point, the thermometer must be immersed in a bath which consists of shaved ice over which water has been poured to remove the air from the interstices. The thermometer must be allowed to stand in the bath until the reading becomes stationary. Particular care must be given to this if the thermometer has just previously been heated. Great care must also be taken that the ice and water are pure, as slight traces of salt will change the melting point. As was the case in the vapor bath and, in fact, in all uses of this type of thermometer, not only the bulb but also the stem must be immersed up to the top of the mercury column. Corrections for the protruding mercury filament are both troublesome and uncertain.

For precise determinations such as are carried out in national bureaus, *e.g.*, in the Bureau of Standards at Washington, many additional precautions are taken to insure standardization of these points. This is essential to avoid all uncertainties in industries.

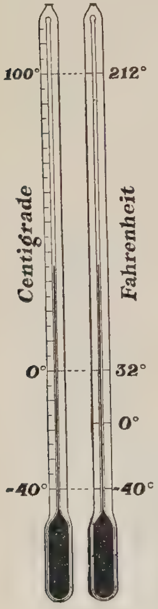


Fig. 137

The stem of the thermometer must now be calibrated to discover all non-uniformities in its bore. This is done by means of a small quantity of mercury which can be shifted along the tube and its length in different positions recorded in terms of the etched scale divisions.

213. Scales of Temperature.—The scale of temperature which is generally in use in continental Europe and which is used in scientific work everywhere is the centigrade scale. It names the two fixed points 0° and 100°, and thus divides the fundamental interval into 100 parts. Other scales have been devised which were originally based on other and less satisfactory fixed points. They have since been restated in terms of our fixed points but naturally they lack the simplicity of the centigrade scale. Most common among these is the Fahrenheit scale which is in general use for domestic and engineering purposes among English-speaking people. In terms of this scale, our fixed points lie at 32° and 212°. The fundamental interval is thus divided into 180 parts, and the Fahrenheit degree is smaller than the centigrade degree in the ratio of 100 to 180 or 5 to 9. To convert the reading on one of these scales to the other, it is necessary to compare the distances from one of the fixed points, remembering that the numbers of degrees in the same interval have the ratio 5:9. Thus, if C. is the centigrade reading and F. the Fahrenheit reading at the same temperature,

$$\frac{5}{9}(F - 32) = C.$$

Of these two scales the Fahrenheit is the older. It was first suggested, in 1700, by the man whose name it bears. Fahrenheit set the zero point at a temperature which he succeeded in getting with a mixture of sal ammoniac and snow. It was approximately the lowest winter temperature in Danzig, where he lived. He arbitrarily set the 96° mark at blood temperature. Somewhat later he put the fundamental points at 32° and 212°, as we now have them. In 1730 Reamur constructed a thermometer, using diluted alcohol as the expanding fluid, and set the transition points of water at 0° and 80°. This scale is still used for domestic purposes in parts of northern Europe. Ten years later, Celsius of Upsala chose the freezing point of water as 100° and the boiling point as 0°. In the same year Linné, also at Upsala, reversed this choice, placing the fixed points where we now have them in the centigrade scale. In continental Europe this scale is usually called the Celsius scale in honor of the man who took the first steps toward establishing it.

214. Special Types.—Most mercury thermometers for ordinary use are constructed to read slightly beyond the fundamental interval. The sensitivity can be increased by decreasing the diameter of the tube or increasing the volume of the bulb. There is a definite limit, however, in both cases. In too fine a tube the viscosity of the mercury prevents rapid flow. If the bulb is too large, it will require too long a time to take up the temperature of its surroundings; and if used to measure the temperature of small bodies, it may take so much heat from them as to sensibly alter their original temperatures.

The clinical thermometer is one type of "maximum" thermometer. It has a constriction just above the bulb. The mercury rises through this on expansion but cannot return until it has been shaken vigorously. It thus records its maximum temperature even after the bulb has cooled.

The low temperature limit of a mercury thermometer occurs at the freezing point of mercury, −39° C. Other liquids may be used at considerably lower temperatures, *e.g.*, pentane, which freezes at − 200° C. Mercury in a closed tube will remain liquid at fairly high temperatures. The pressure, however, becomes very high, and glass cannot be used as a container above 550°, the temperature at which it begins to soften. Quartz thermometers may be used up to 750° C.

For very high and very low temperatures, other types of thermometers must be used. The gas thermometer has a wide range and follows the thermodynamic scale closely but is not convenient for ordinary work. Thermoelectric couples, electric resistance thermometers and radiation pyrometers make use of different properties of materials, and will be described when we become familiar with the subject of Electricity.

PROBLEMS

1. Find the temperature on the centigrade scale when a Fahrenheit thermometer reads 72°; also when it reads − 72°.

2. What Fahrenheit reading corresponds to a temperature of 60°C.? To −20°C.?

3. At what temperature do the Fahrenheit and centigrade thermometers have the same readings?

4. The zero of the gas scale is at −273°C. Find the corresponding reading on the Fahrenheit scale.

CHAPTER XV

EXPANSION

215. Linear Expansion.—One of the most striking and best-known manifestations of change of temperature in bodies is the phenomenon of expansion. We have already made use of this as a practical means of measuring temperature itself. That alone would require us to analyze the phenomenon in more detail. All three forms of matter—solid, liquid and gas—usually increase their dimensions on heating. This is, however, not universally true; and even where it is true, the extent to which the increase occurs differs widely from one material to another and for the same material in different parts of the temperature scale.

The change of most quantities which characterize the state of a body and which vary with the temperature, may be described in terms of a "temperature coefficient." This is defined as the ratio of the change of the quantity per degree change of temperature to the original value of the quantity. For example, if a bar of iron is heated, its length increases. Its coefficient of linear expansion is the change of length per degree rise of temperature divided by the original length. We shall confine ourselves to the coefficient which takes the original dimension at 0° C. The coefficient of linear expansion may then be written:

$$\beta = \frac{L_2 - L_1}{L_o(t_2 - t_1)},$$

where t_2 is the higher temperature and L_2 the corresponding length, t_1 the lower temperature and L_1 the corresponding length, L_o the length at 0° C. β is thus a property of a substance, not of a particular body.

If β is the same for all parts of the temperature scale, the body is said to expand uniformly with the temperature. If this is not the case, we obtain for β a mean coefficient which must be more precisely defined. The most useful of the mean coefficients is that taken between 0° and the temperature in question. In this case we may write, since $t_1 = 0$,

$$\beta = \frac{L_t - L_o}{L_o t};$$

or, solving for L_t,

$$L_t = L_o (1 + \beta t).$$

It is this temperature coefficient which we shall use in the remainder of this course.

Methods of measuring this coefficient require careful maintenance of a uniform temperature throughout the body, and its accurate determination. The change of length produced by a moderate change of temperature is for most solids very small. This must be measured with precision—for example, by focusing a microscope on a definite line ruled on the body and measuring by means of a micrometer screw the amount necessary to displace the microscope in order to refocus after expansion. L_o is defined as the length at 0° C., but if the expansion is small, the length at room temperature may be substituted for it without materially changing the resulting value of the expansion coefficient. This would not generally be possible for liquids or gases where the expansion per degree is much greater.

An isotropic body is one which possesses the same properties in all directions. Such a body would have only one coefficient of linear expansion which would apply equally to its length, width and height. There exist, however, bodies which are not isotropic, such as crystals, in which expansion takes place differently along different axes. Such a body would have, in general, three different linear expansion coefficients.

216. Volume Expansion.—Another important expansion coefficient, not only for solids but also for fluids, is the volume coefficient. This is defined as the ratio of the change of volume per degree rise in temperature to the volume at 0° C., or, as a formula,

$$\alpha = \frac{V_t - V_o}{V_o t}.$$

There is a simple relation between the coefficient α and β for a given material. Suppose we consider an isotropic body in the shape of a cube with original sides L_o and original volume $V_o = L_o^3$. After a rise of temperature to t, the length of the sides will be L_t and the new volume $V_t = L_t^3$. Now since

$$\beta = \frac{L_t - L_o}{L_o t},$$

solving for L_t,

$$L_t = L_o(1 + \beta t).$$

Cubing both sides of the equation,

$$L_t^3 = L_o^3\,(1 + 3\beta t + 3\beta^2 t^2 + \beta^3 t^3).$$

If β is small, we may neglect the last two terms which contain the square and cube of β. Putting in also the symbols for volume, we have

$$V_t = V_o(1 + 3\,\beta t),$$

or

$$\frac{V_t - V_o}{V_o t} = 3\beta = \alpha.$$

For small values of β we may therefore, for isotropic bodies, set the volume coefficient equal to three times the linear coefficient. The student can determine what the relation would be if the body possessed three independent coefficients, β_1, β_2, β_3, along three different axes at right angles.

217. Expansion in Containers.—When a homogeneous solid changes volume by expansion due to a uniform rise of temperature throughout its material, it is a matter of experience that no new stresses are set up within the body. This is equivalent to saying that the space available for any interior part of the body changes its shape and dimensions precisely as does the interior part itself. If this were not the case—if, for example, the space occupied by the small element increased less rapidly than the element itself—new stresses would necessarily appear at the higher temperature. We may conclude, therefore, that a cavity in a body expands at the same rate as a body of the same material. This is of great importance in calculating the change in content of a vessel. To find how much the content of a flask will increase with a given rise of temperature, we treat its cavity as if it were made of the material of the flask.

In the measurement of the expansion of a liquid by observing the amount it overflows from a container or, as in the case of a thermometer bulb, the amount it rises in a connecting tube, corrections for the expansion of the container must be applied. Suppose, for example, we have a container made of a material whose volume coefficient is α_s, filled at 0° C. with a liquid whose volume coefficient is α_L. At the temperature t, the content of the flask is

$$V_s = V_o + V_o\alpha_s t,$$

and the volume of the liquid,

$$V_L = V_o + V_o\alpha_L t.$$

The volume of the liquid which overflows at the higher temperature is therefore

$$V_L - V_s = V_o\,(\alpha_L - \alpha_s)t.$$

$\alpha_L - \alpha_s$ is called the apparent coefficient of expansion of the liquid in this container. If $\alpha_s = \alpha_L$, the liquid will continue to fill the container but will not overflow as the temperature rises. For the liquid to overflow with rising temperature, α_L must be greater than α_s.

In many cases the expansion of a liquid is so slight that it is not easy to detect its change in volume when enclosed in a container which itself undergoes expansion. A very ingenious device permits one to compensate for this. If, for example, one should introduce into the liquid a body which changes volume at just the same rate as the container, the rise of the liquid in a connecting tube would depend only on its own expansion coefficient. For example, the expansion coefficient of glass is about .00002 per degree centigrade and that of air is .00366, 183 times as great. If, therefore, one should introduce into a glass flask full of liquid a volume of air equal to 1/183 of the volume of the flask, its expansion would exactly compensate for the expansion of the glass. This may be done, as shown in Fig. 138, by a tube containing air, open at the bottom and closed at the top by a stopcock which permits adjustment. The expansion coefficient of a gas is only constant when the pressure is constant. As the liquid rises in the tube the pressure on the enclosed air is increased. Therefore too large a range of expansion must not be used with a single adjustment.

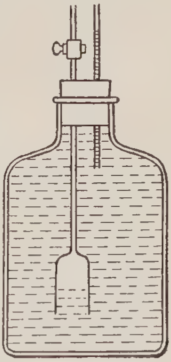


Fig. 138

218. Determination of Expansion of Liquids.—If one knows the volume coefficient of a container and can measure precisely the amount of liquid which overflows, it is possible by this means to obtain the coefficient of expansion of the liquid. Or if one knows the expansion coefficient of the liquid, the expansion coefficient of the container may be determined by this means. Further, if both coefficients are known, one may determine the expansion of a new solid by immersing a sample in the liquid and measuring the overflow on heating as before. All original volumes must, of course, be determined.

The student can show how the coefficient of expansion of the immersed solid may be determined from the data of the operation just described.

A celebrated method for determining the expansion of a liquid independent of the container is due to Dulong and Petit, French physicists, who worked about a hundred years ago. The liquid to be investigated is placed in two communicating tubes, as shown in Fig. 139. The tubes are then brought to the temperatures t_o and t, and the heights h_o and h_t noted.

Since the liquid is in equilibrium, the pressures in the horizontal tube are equal on the two sides, and we may write

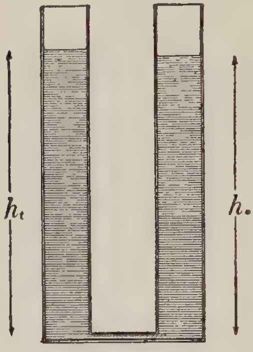


FIG. 139

$$h_o d_o = h_t d_t,$$

or

$$\frac{h_t}{h_o} = \frac{d_o}{d_t}.$$

We must now restate the formula for α in terms of densities.

$$V_t = V_o(1 + \alpha t);$$

but since

$$V = \frac{M}{d},$$

$$\frac{M}{d_t} = \frac{M}{d_o}(1 + \alpha t);$$

and therefore

$$\frac{d_o}{d_t} = 1 + \alpha t = \frac{h_t}{h_o},$$

$$h_t = h_o\,(1 + \alpha t).$$

Since the cross-section of the tubes has not entered the calculation, we see that the result is independent of the rate of expansion of the container.

It may seem paradoxical that this experiment should give the correct result, even though one of the containers increases its cross-section with rising temperature. It is easy to see, however, that, as the temperature rises on one side, the liquid will flow through the horizontal tube. If we let A_o and A_t equal the cross-sections of the cold and hot cylindrical containers respectively, we may write the masses of liquid as $A_o h_o d_o$ and $A_t h_t d_t$. $h_o d_o$ must remain equal to $h_t d_t$, since this determines pressure equilibrium. The ratio of the masses of liquid on the two sides must therefore change with changing A. Thus, if

A_t increases and A_0 remains constant, liquid must flow from the cold cylinder to that which is being heated. h_0 will decrease and must be read after the second cylinder is heated, and not before. It is this flow of liquid which compensates for the expansion of the container.

219. Expansion of Water.—It has long been known that water in its liquid state possesses extraordinary properties of expansion. The density not only fails to increase uniformly as the temperature is lowered, but reaches a maximum in the neighborhood of 4° C., and below that temperature decreases. Fig. 140 is a curve which shows the change of volume of a given mass of water with change of temperature. The position of minimum volume (maximum density) is very close to 4° C. when temperatures are determined with a mercury thermometer. The curve is fairly symmetrical on both sides of this point as far as measurements are possible.

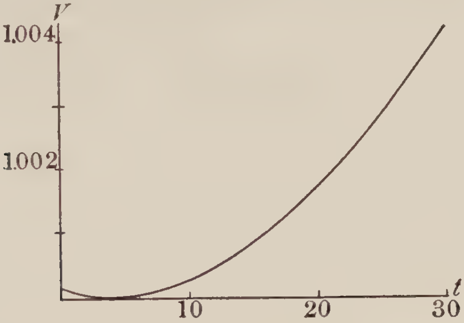


FIG. 140

If one tried to follow these small changes of volume at low temperatures by enclosing the water in a glass flask and watching the surface in a connecting tube, the expansion of the flask would probably mask the small effects. It is easy to compensate for the expansion of the container with some such device as that described in §217.

In large masses of water, where losses of heat occur mostly at the surface, the colder water sinks to the bottom until the whole mass attains a temperature of 4° C. After that, the surface cools further and freezes without altering the temperature of the water below.

220. Expansion of Gases.—In the case of gases, we have again only volume expansion. The volume of a gas depends, however, not only on the temperature but also on the pressure. This is also true for solids and liquids, but, as has been learned in Mechanics, the coefficient of elasticity is far greater for solids and liquids than for gases. This signifies that moderate changes of pressure produce exceedingly small changes in volume. Before one undertakes to determine the coefficient of expansion of a gas, one must first decide how the pressure will change during the operation of heating. The simplest case, and the one of greatest importance, is that in which the pressure is kept constant.

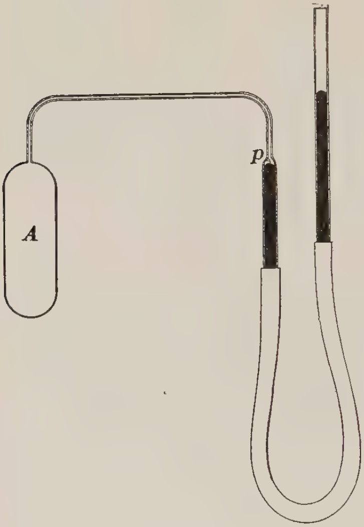


Fig. 141

To measure this coefficient, for which we shall use the symbol α_v, the gas is enclosed in a container which may be heated. It is separated from the outside air by mercury in an adjustable U tube, as in Fig. 141. At the new temperature the mercury tube is adjusted until it indicates the original pressure. The corresponding volume is recorded. Careful corrections must be made for the gas in the tube which emerges from the temperature bath. Changes of volume of the container are so much smaller than those of the gas that they can usually be neglected. High precision will require a correction for this also.

221. Law of Gay Lussac.—The results of careful measurement of the expansion coefficient of gases are recorded in the Law of Gay Lussac, which states that α_v has the same value for a large group of gases. This group includes the so-called permanent gases and, in general, those which are far from the liquefaction point and which in our discussion of Boyle's Law were called "perfect." The value of this coefficient is 0.003665, or 1/273 per degree. It is independent not only of the temperature at which it is measured but also of the pressure. In other words, if the temperature of a perfect gas is raised one degree centigrade under constant pressure, it will increase its volume by 1/273 of the volume which it has at 0° C. under the same pressure.

Gases which are not perfect depart correspondingly from this law. In fact, we may say that gases are more nearly in accordance with this law the more nearly they approach the perfect state, *i.e.*, the greater their temperatures and volumes. It has already been pointed out in the discussion of Boyle's Law that we believe the perfect gases to be a decidedly simple form of matter. The molecules are so far apart that they occupy an extremely small fraction of the volume of the gas, and each molecule spends an extremely small fraction of its time near enough to its neighbors to be under the influence of their mutual attractions or repulsions. Now it is by means of these very intermolecular forces that materials display some of their most characteristic properties. The perfect gases are to a considerable extent free from the individual physical peculiarities which would distinguish the same materials in other physical states. It is natural therefore to believe that the perfect gases, sharing as they do the same value of expansion coefficient, would form very suitable thermometric materials. This is indeed the case, though we must not forget that the perfect gas is an ideal which we approach but do not realize. Under ordinary conditions, even hydrogen shows a measurable departure from Boyle's Law and the Law of Gay Lussac.

222. Pressure Coefficient.—There is another temperature coefficient which should also be constant for all gases which obey the laws of Boyle and Gay Lussac, and in this respect hydrogen behaves very satisfactorily. This is called the pressure coefficient or the temperature coefficient of increase of pressure. We shall use for this the symbol α_p and define it as the ratio of the change of pressure per degree rise in temperature to the pressure at 0° C., the operation being carried out at constant volume. The equation is

$$\alpha_p = \frac{p_t - p_o}{p_o t}.$$

This again defines the mean coefficient between 0° and t°. A device similar to that used for measuring α_v may also be employed to measure α_p. The mercury column must now be adjusted so that the volume of gas is kept constant. The resulting change in the difference in heights of the two mercury columns is recorded as a measure of the pressure change. Very precise measurements of this coefficient show not only that it is nearly constant for any permanent gas, but that for all of them it approaches very closely their common value for the volume coefficient, viz., 0.003665 per degree centigrade.

223. Volume and Pressure Coefficients.—That the pressure and volume coefficients of a perfect gas are equal can be shown to be a consequence of the laws of Boyle and Gay Lussac. If we suppose a certain mass of gas heated from 0° to t° at constant pressure p_o, we may write

$$v_t = v_o(1 + \alpha_v t).$$

Multiplying both sides by p_o,

$$p_o v_t = p_o v_o(1 + \alpha_v t).$$

$p_o v_t$ is the product pv at the new temperature t. According to Boyle's Law, this product has a fixed value at a definite temperature. The equation may therefore be written

$$(pv)_t = (pv)_o\ (1 + \alpha_v t). \tag{1}$$

Beginning again at temperature 0° and pressure and volume p_o and v_o, and again raising the temperature, now keeping the volume constant and allowing the pressure to change, we have

$$p_t = p_o\ (1 + \alpha_p t).$$

Multiplying by v_o,

$$p_t v_o = p_o v_o (1 + \alpha_p t);$$

and, for the same reasons as before,

$$(pv)_t = (pv)_o (1 + \alpha_p t). \tag{2}$$

Equations (1) and (2) can both be true only if $\alpha_p = \alpha_v$. We may therefore for perfect gases omit the subscript and write both coefficients α.

224. Absolute Temperature and the Gas Law.—We see from these equations that for perfect gases the product pv increases with the temperature at a uniform rate, regardless of the individual changes of p or v. This means that differences in pv are proportional to the corresponding temperature differences. It is more convenient to alter the temperature scale so that we may say that pv is proportional to this new temperature. For this to be true, the zero of the temperature scale must be shifted so that $pv = 0$ when the new temperature is zero. In the equation

$$(pv)_t = (pv)_o (1 + \alpha t),$$

$(pv)_t = 0$ if $1 + \alpha t = 0$; that is, if $t = -\frac{1}{\alpha}\cdot$ Now α is very nearly $\frac{1}{273}\cdot$ Therefore the zero of the scale must be shifted to −273° C. This is called the absolute zero, and temperatures counted from it are called absolute temperatures. Ice melts at 273° absolute.

Representing absolute temperatures by the symbol T, we have

$$t \doteqdot T - 273 = T - \frac{1}{\alpha},$$

and substituting in the above equation,

$$(pv)_T = (pv)_o \alpha T.$$

Replacing $(pv)_o\alpha$ by R,

$$pv = RT,$$

the subscript T being no longer necessary to distinguish $(pv)_T$ from $(pv)_o$.

This change of scale may be seen graphically in Fig. 142. If one plots the product pv against centigrade temperature, the graph is a straight line with slope $(pv)_0\alpha$ which intersects the temperature axis at $t = -\frac{1}{\alpha}$. If the origin is moved to this intersection point, the curve becomes a straight line through the origin. The coördinates are now proportional to each other, and the proportionality factor is the slope $(pv)_0\alpha$.

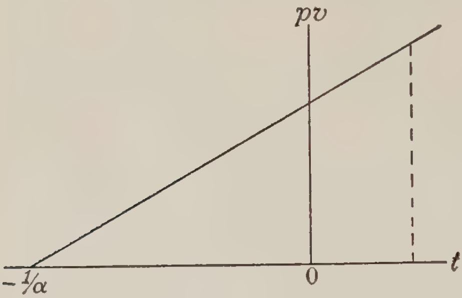


FIG. 142

The equation

$$pv = RT$$

is called the Gas Law. It contains the two laws of Boyle and Gay Lussac from which it was derived.

If v is taken as the volume occupied by one gram, R has a different value for each gas. It is often more convenient to define v as the volume occupied by one mol, *i.e.*, a number of grams of the gas equal to its molecular weight. Avogadro's Law tells us that under the same conditions of pressure and temperature equal volumes of all perfect gases contain the same number of molecules. That is equivalent to saying that under similar conditions equal volumes will contain masses proportional to the molecular weights. Therefore for a given pressure and temperature, one mol of any perfect gas will occupy the same volume. It follows that if v represents this volume, R must have the same value for all gases. Its value is 8.315 joules per degree. If V is the total volume of gas in a given problem and n the number of mols, the gas equation may be written

$$pV = nRT.$$

225. Standard Hydrogen Thermometer.—In our discussion of thermometers, the fact was recognized that mercury was chosen as a thermometric substance because it was so convenient, and not because we had any reason to believe that by assuming its expansion to be uniform, we should make our division of the scale a particularly good one. On the contrary, it was recognized that in other respects mercury had properties that were not at all simple. We have now studied a state of matter into which all matter may be put at high temperatures and low pressures, and in which state all substances share the important properties of obeying Boyle's Law and possessing the same expansion and pressure coefficients. The perfect gas is therefore chosen as our standard thermometric substance.

The standard thermometer consists of a container filled with hydrogen which is heated at constant volume. The pressures are then carefully measured. The fixed points are kept as before, melting ice as 0° and condensing steam under 760 mms pressure as 100°, or, as we may now say, as 273° and 373° absolute. If $p_\circ$ is the pressure when the gas is immersed in melting ice, any other pressure p will correspond to an absolute temperature T, given by the equation

$$\frac{T}{273} = \frac{p}{p_\circ}.$$

Since considerable skill is required to manipulate this type of thermometer, and since its construction is complicated and inconvenient, it is rarely used in practice but rather as a standard with which mercury thermometers for precise use are compared. When each division of a mercury thermometer has been compared with the hydrogen thermometer, it is said to have been calibrated, and its readings may then be converted into standard temperatures.

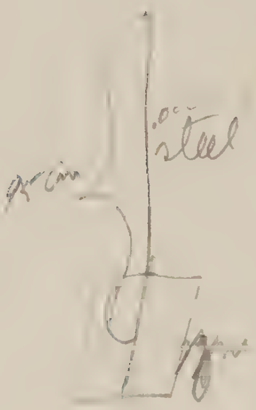
PROBLEMS

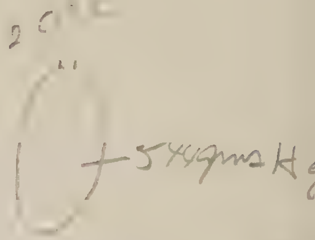
1. A steel bridge 200 ft. long will change how much in length when the temperature changes from − 20° C. to + 20° C.? Coefficient of linear expansion of steel is .000012.

2. A gas has a volume of 500 c.c. when it is under a pressure of 60 cm of mercury and at a temperature of 27° C. Find its volume under standard conditions.

3. A liter of air under standard conditions weighs 1.293 grams. Find the density of air if the pressure is 5 atmospheres and the temperature 30° C.

4. A steel bar 2 cm² in cross-section is so placed that its two ends are absolutely fixed. How much force would it exert for each degree rise in temperature? Coefficient of expansion is .000012. See table for Young's modulus on page 103.





final temp 100

? wt of Hg

$V_{gt} = V_{g0}(1 + \alpha_g t)$

$V_{Lt} = V_{L0}(1 + \alpha_L$

$V_{Lt} - V_{gt} = V_0 \alpha_L t - V$

$= V_0 t(\alpha_L - \alpha_g$

$\frac{M}{V_t} = \frac{M}{V_0}(1 + \alpha t$

CHAPTER XVI

CALORIMETRY

226. Quantity of Heat.—Calorimetry is concerned with the measurement of the quantity of heat. In order to express the results of these measurements, a unit of heat must first be defined. In the metric system the unit is defined as that quantity of heat which will raise the temperature of one gram of water from 15° to 16° C. This unit is called the calorie. The unit of heat used in engineering practice in English-speaking countries is the British Thermal Unit (B.T.U.). It is defined as the amount of heat required to raise the temperature of a pound of water one degree Fahrenheit.

The quantity of heat necessary to change the temperature of a certain mass of material one degree is a characteristic property of the material, and is described as its thermal capacity. It varies widely with the material and even differs from one temperature to another for the same material. The thermal capacity of a substance is defined as the number of calories per gram per degree required to raise the temperature of that substance. Thus, the thermal capacity of water at 15° is 1 calorie per gram per degree. Quite as frequently the term specific heat is used. This bears somewhat the same relation to thermal capacity as specific gravity does to density. The specific heat of a material is the ratio of its thermal capacity to that of water at 15°. It is therefore numerically equal to the thermal capacity, but like specific gravity is a pure number.

Fig. 143 shows how the specific heat of water varies with the temperature. The variation is very slight, and for most purposes the specific heat may be considered constant. A minimum value occurs in the vicinity of 40°. The specific heat at 15° is very close to the average value. In other words, a calorie is 1/100 of the heat required to raise one gram of water from 0° to 100°.

227. Method of Mixtures.—Very ingenious methods have been devised for the difficult task of measuring quantities of heat. One of the chief difficulties consists in the prevention of losses to outside bodies by conduction or radiation. Methods which avoid these most successfully involve for the most part transformations of energy of other types, *e.g.*, electrical, into heat. We must defer the discussion of these until later. One method, however, of great historical and practical interest involves only principles already familiar to us. This is the method of mixtures. It consists in placing in thermal contact two bodies which are originally at different temperatures and observing the change of temperature which each undergoes. Believing as we do that heat is a form of energy and that energy is conserved, then if, during the process, there is no energy exchanged with the outside, either by heat flow or by mechanical work done by expansion, the amount of heat given up by the hotter body must be equal to that gained by the colder. Since the thermal capacity of a substance is the heat necessary to change the temperature of one gram of it one degree, the amount of heat necessary to raise the temperature of a body whose mass is m by $(t - t_1)$ degrees must be $mc(t - t_1)$ calories, c being the thermal capacity of the material. Similarly, the heat given off by the second body when its tem-

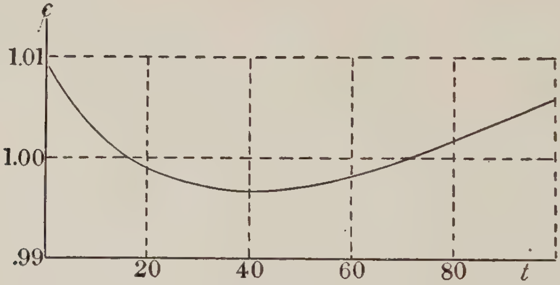


FIG. 143

perature falls from t_2 to the same final temperature t, is $m'c'(t_2 - t)$.

Equating, $$mc(t - t_1) = m'c'(t_2 - t).$$

If one of the bodies consists of more than one material, additional terms must enter the equation, one for each material in the hotter body and one for each material in the colder. We may then from this equation determine any one of these quantities, if we know all of the others. In particular, the specific heat of one of the materials may be found by this method. The materials used in this method may be either solids or liquids. It is, in fact, advantageous to have at least one liquid present, since thermal contact is then made more easily, and by stirring we attain the final temperature more quickly.

A container, properly insulated and sometimes containing a suitable liquid into which the material to be investigated is placed, is called a *calorimeter*. The measurement of quantities of heat by this method, or others similar to it, is called *calorimetry*.

The method of mixtures is sometimes used to determine high temperatures not accessible to ordinary thermometers. For example, the temperature of a furnace may be found by holding in it a bit of metal with a high melting point and then measuring the amount of heat the metal loses when brought to a lower temperature in a calorimeter. The specific heat of the metal must be known, as well as the constants of the calorimeter.

228. Law of Dulong and Petit.—If the specific heat of a chemical element is multiplied by its atomic mass, the result will no longer be characteristic of the mass of the substance but rather of a definite number of its atoms. This quantity is called the *atomic thermal capacity* of the substance. The French scientists, Dulong and Petit, found that all chemical elements in a solid state have nearly the same atomic thermal capacity. The average value is very close to six calories per degree. The law does not hold exactly, but appears to be more nearly true at higher temperatures. Modern experimental research has extended measurements of specific heat both to lower and higher temperatures, and all elements appear to follow the typical curves shown in Fig. 144. The curves all seem to start from zero at absolute zero temperature and to rise gradually to the predicted value of six calories. At ordinary temperature, most substances have reached this fairly steady behavior. Carbon and boron have been notable exceptions in this respect. We know now that their behavior is not essentially different if sufficient temperature range is considered.

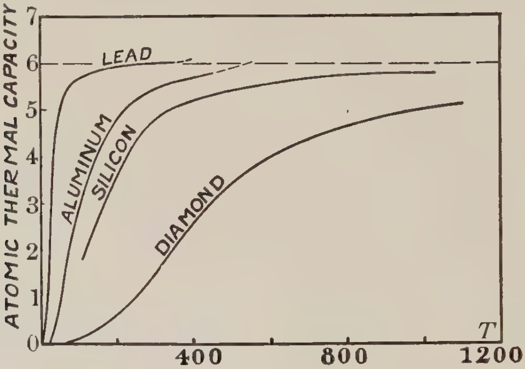


FIG. 144

Modern theory has been able to account for the failure of all materials to obey the law at sufficiently low temperatures. It is highly important to know that the atoms of all solids differ so little in the way in which they store the energy which they receive on rise of temperature.

229. Specific Heat of Gases.—In the case of the specific heat of a gas, it is necessary to make the definition more precise. In the study of the expansion of gases it was noticed that the volume of a gas depends not only on the temperature but also on the pressure. Therefore it was necessary to state the pressure changes which were to take place before the expansion coefficient could be defined. This was not necessary for liquids and solids because changes of pressure produced negligible changes of volume. Bodies, in general, expand on heating. Work is done during the expansion in pushing back the surrounding medium, and the amount of work depends on the pressure which the medium is exerting on the body. If, for example, a body increases its volume by v under a pressure p, we have learned in Mechanics that the work done is pv. Therefore, when a body is heated, energy is required both to raise the temperature and to do the work of expansion. For solids and liquids the changes of volume are so small that the work of expansion is usually negligible. For gases, however, it is a fairly large quantity, and since it depends on the pressures under which the gas is expanding, those pressures must be determined before the specific heat can be defined. In other words, the gas has an infinite number of specific heats corresponding to the various manners in which the pressure may change during rise of temperature. Two of these are of particular interest. They are the specific heat at constant volume c_v, and the specific heat at constant pressure, c_p.

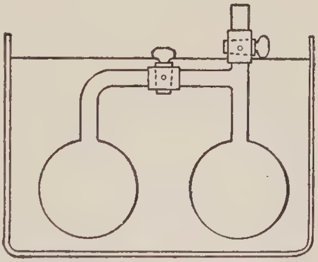


FIG. 145

230. Free Expansion.—In order to understand clearly the relation between the specific heats of a gas, it is necessary to study the results of an experiment which was first carried out with precision by the English physicist, Joule. Two cylinders connected by a tube with a stopcock were placed in a calorimeter, as shown in Fig. 145. One cylinder contained air at about 2.2 atmospheres pressure; the other was evacuated. On opening the stopcock between them, the air rushed into the evacuated cylinder until pressures were equal. Careful readings of the temperature of the calorimeter were taken before and after the experiment, but no appreciable change of temperature could be detected. We now know that for ordinary gases near their liquefaction point a slight cooling effect takes place, but that for perfect gases this effect vanishes. In this experiment no external work has been done by the gas on expansion. The temperature of the gas is determined by the average kinetic energy of the molecules. If the molecules attracted each other, it would require work to separate them to the greater distance from each other which is necessary at the larger volume. This must be done at the expense of the kinetic energy, and consequently the temperature would be reduced. We must conclude, therefore, that the molecules of a perfect gas are already separated by distances so great that the forces of attraction are negligible. Any further increase of volume will take place at constant temperature unless external work is done, as in pushing back a piston. In other words, the thermal energy of a perfect gas does not depend on the volume but on the temperature alone. For non-perfect gases internal work must be done on expansion, and the energy required when the volume is increased without change of temperature is greater than that used in doing the external work.

231. Work Done During Expansion.—If the volume is kept constant, no external work is done, and the heat given to the gas increases only the energy of its thermal agitation. If the pressure is kept constant, the volume increases and external work is done. Therefore we shall expect c_p to be greater than c_v. For ordinary gases the relation between c_p and c_v is complicated, and one of them cannot in general be calculated from the other even when the expansion coefficient is known. But in the case of perfect gases the relation is quite simple. We know from experiment that, if a perfect gas has its temperature raised 1° under any pressure condition, its change of internal energy of agitation is always the same. The internal energy depends only on the temperature of the gas, and not on the volume it occupies. We may therefore say in this case that $c_p - c_v =$ external work per gram per degree done in expansion.

To calculate this external work, it is only necessary to multiply the constant pressure by the change of volume. For the original temperature,

$$pv_o = RT,$$

where R is the universal gas constant and v_o is the volume occupied by one molecular weight. For the final temperature which is one degree higher,

$$pv_1 = R(T + 1),$$

p being kept constant. Subtracting,

$$p(v_1 - v_o) = R.$$

The universal gas constant R is therefore equal to the work done in the expansion of one mol of a perfect gas during the rise of temperature of 1° under constant pressure, or

$$R = M(c_p - c_v).$$

The ratio $\frac{c_p}{c_v} = \gamma$ is also an important quantity, but it has not a universal value even for all perfect gases. It depends on the structure of the molecule. For example, for most monatomic perfect gases, $\gamma = 1.6$, and for diatomic perfect gases, $\gamma = 1.4$. A knowledge of this ratio is necessary in order to determine the adiabatic coefficient of elasticity, which was used in computing the velocity of sound through gases.

PROBLEMS

1. Find the temperature of an oven as determined by the following data: A piece of platinum weighing 120 grams, having a specific heat of .033, is put into the oven, and after it has attained the temperature of the oven it is taken out and dropped into an aluminum cup weighing 60 grams and having a specific heat of .21. The cup contains 200 grams of water at temperature 16°C. The final temperature of the water is 27°. Find the temperature of the oven.

2. A block of metal of 2,000 gms. mass is heated to a certain temperature and then immersed in 1,000 gms. of water at 0°C. The temperature of the water rises to 15°C. When a block of the same metal of 1,000 gms. mass is heated to the same temperature and then immersed in 1,000 gms. of water at 0°C., the temperature rises to

10°C. Find the temperature to which the blocks were raised, and the specific heat of the metal.

3. How many calories are equal to one B. T. U.?

4. A small block of metal heated to 200°F. and then dropped into 0.2 lb. of water at 50°F. in a copper (specific heat, .095) calorimeter of 0.2 lb. mass produces a final temperature of 70°F. When the experiment is repeated with 0.2 lb. of glycerine, the final temperature becomes 80°. Find the specific heat of glycerine.

CHAPTER XVII

CHANGE OF STATE

232. Fusion.—In general, a substance may exist in three states, as solid, liquid or gas. There are cases in which the number of states may be greater than three, as, for example, the case where a substance, like sulphur, may exist in different crystalline forms. We shall confine ourselves for the present to the simpler case, and study the phenomenon of transition from one of these three states to another.

If a solid is heated its temperature will rise until it reaches a certain value, at which the solid passes into a liquid state. If the solid is crystalline this change will be abrupt. The temperature at which the liquefaction begins will be a definite one, and as more heat is applied the temperature remains fixed until the solid is completely liquefied. Further addition of heat will then again cause rise of temperature. If the liquid is cooled it will begin to solidify at the same temperature, and again the temperature will remain constant, in spite of the withdrawal of heat, until the solidification is complete. Solids which are not crystalline but amorphous have no definite melting point. For them the transition is gradual instead of abrupt. They gradually become plastic as the temperature rises, until the solid state merges imperceptibly into the liquid state. Such solids can hardly be said to have changed state at all. In fact, amorphous solids may be thought of as liquids of very great viscosity. In the case of ice, the melting point is so sharply marked that we have been able to use it as a fixed point of our thermometric scale.

During the process of fusion, solids usually experience a change in volume. The majority of materials expand on melting. The solid has a greater density than the liquid and sinks when immersed in it. Ice forms a notable exception and contracts on melting. Ice therefore floats on water. This abnormal behavior of water is in some respects beneficial and in some respects destructive. That winter ice forms and remains at the top of masses of water insures its rapid and complete fusion in the spring. The same property, however, results in the splitting of rocks and trees and the bursting of plant cells or waterpipes when the water freezes. The density of ice at 0° is very nearly 0.9 grams per cc. This is a change of volume of about 10 per cent. Materials in general change less than this. Lead contracts on freezing by 3 per cent, mercury by slightly more than 3 per cent. Bismuth expands on freezing, likewise, by about 3 per cent.

233. Effect of Pressure on Freezing.—As usual, for thermal phenomena in which no gases take part, the effect of pressure on melting and freezing is comparatively small. It is, nevertheless, of great importance. When we made the statement that the melting point of ice shall correspond to 0° of the centigrade scale, it was implied that the change of state should take place under atmospheric pressure. Small variations from this produce hardly measurable effects in the melting point. If the pressure is increased to ten atmospheres, the melting point falls to $-0.075°$.

A lowering of melting point with increase of pressure is characteristic of those materials which expand on freezing. We may think of this phenomenon in somewhat the following way. Suppose we have a solid under one atmosphere pressure and slightly under its normal melting point. We know it cannot liquefy under these conditions. If now the pressure is increased, this tends to decrease the volume. If the liquid state has a smaller volume than the solid, the increased pressure will assist the process and melting may actually take place at the lower temperature. If the liquid state has the larger volume, a diminution of pressure will lower the melting point for similar reasons.

Fig. 146 is a melting point curve for a substance, such as water, which expands on freezing. The coördinates are pressure and temperature, and each point of the curve indicates a pressure and corresponding temperature at which the solid will melt without change of temperature if heat is applied. We frequently say that the curve is made up of points at which the solid and liquid may coexist. At points not on the curve we cannot have both states at once. For points on the right, we have liquid only. For the points on the left, solid only. Curves of this sort are used to represent the transition points for all changes of state. They are most useful in discussing the behavior of the material. For example, if a point is taken slightly to the left, in which case we have the solid state, we may reach the melting point curve by moving vertically upward; that is, ice may be liquefied by increasing the pressure without changing the temperature. This explains the slipperiness of ice just below the freezing point. Pressure liquefies it and the water acts as a lubricant. It also accounts for the apparent plasticity of ice. Any endeavor to change the shape of ice slightly by pressure liquefies it where the pressure is applied. On removal of the pressure it freezes again in the new form.

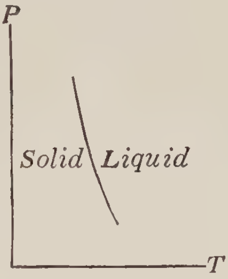


Fig. 146

An interesting experiment illustrates this property. A block of ice is supported at its ends and holds a heavy iron weight by means of a wire which encircles the ice, as shown in Fig. 147. The weight of the iron is distributed over the small area of the wire and consequently produces a high pressure directly underneath. At these points the ice is melted and the wire is pulled down through the ice. As the wire proceeds the water above it is released

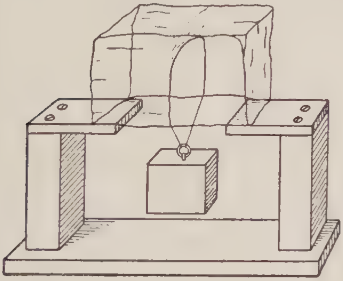


Fig. 147

from the high pressure and is robbed of heat by the process of melting which goes on just below it. As a consequence, it freezes again and seals the block of ice together in its original shape.

234. Freezing Mixtures.—When water contains a salt in solution its freezing point is lowered. Solid ice and undissolved salt cannot coexist at 0° C. If they are mixed they endeavor to form a liquid solution. This requires the melting of the ice, during which heat is withdrawn from the mixture and its surroundings and the temperature is lowered. It is not difficult to reduce the temperature to 0° F. by means of this freezing mixture.

235. Heat of Fusion.—The amount of heat which is necessary to convert the solid into the liquid state while its temperature is stationary at the melting point is likewise a definite quantity. The number of calories which are required to melt one gram of a solid at the melting point is called the heat of fusion. A precisely similar amount must be withdrawn from the liquid in order to solidify it under the same conditions. For ice the heat of fusion is 80 calories per gram. An exact knowledge of this quantity allows one to use the process of change of state in calorimetry. For example, by a method of mixtures the thermal capacity of a body may be determined by measuring the amount of ice it will melt in cooling to 0°. A device for accomplishing this is called an ice calorimeter.

The heat of fusion of ice is higher than that of other materials. Aluminum also has a very high heat of fusion, about 77 calories per gram. Iron requires 30 calories per gram, copper 42, sulphur 10, lead 5, mercury 3. It is notable that so common a material as water should have exceptional values for most of its physical constants.

236. Vaporization and Vapor Tension.—The change from the liquid into the gaseous state has much in common with the transition just studied. The presence of a gas, however, changes the details of the problem; consequently we have in the present change of state some phenomena which we do not meet elsewhere.

It is true here, as before, that the two states may coexist in equilibrium and the transition go on without change of temperature only under very definite conditions of temperature and pressure. If we plot a curve similar to the melting point curve, we find that for all substances it slopes in the same direction, that is, in such a way as to indicate that increased pressure raises the transition temperature. This is to be expected from what has been said before. The gaseous state always has a larger volume than the liquid. If, then, we start with liquid just below the vaporization point, it will be necessary to decrease the pressure and so tend to increase the volume in order to bring the transition point nearer. A curve made up of these transition points with P and T as coördinates is commonly called a vapor tension curve. By vapor tension is meant the pressure of vapor which is just about to condense and which may exist in equilibrium in the presence of its liquid. This term is to be distinguished from the more general one, *vapor pressure.* Vapor pressure indicates the pressure of a vapor under any conditions. For each temperature we have a definite vapor tension, as shown in the curve above. A vapor which is in equilibrium with its liquid is called a saturated vapor. Vapor tension is therefore the vapor pressure of a saturated vapor. Non-saturated vapor is sometimes called superheated. This signifies that the temperature is too high for it to condense at that pressure.

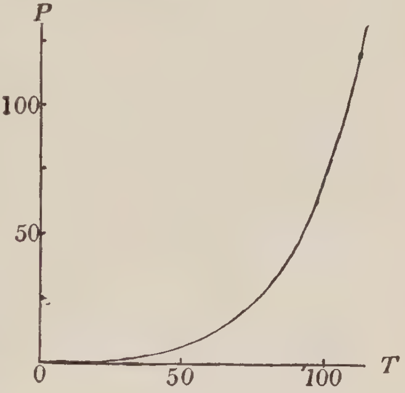


FIG. 148

The situation is unchanged if other gases are present in addition to the vapor in question. The pressure in which we are primarily interested is what is called the partial pressure of the vapor, that is, the pressure the vapor would exert if alone, the other gases being removed. For example, the vapor pressure of the water in the atmosphere is that part of the atmospheric pressure which is due to the water vapor alone. It is usually a small fraction of the total pressure. If the partial pressure of the vapor is lower than the vapor tension corresponding to that temperature, the liquid will continue to evaporate even though the total pressure of the gases exceeds the vapor tension. The presence of the other gases changes the vapor tension curve by an exceedingly small amount. Usually the water vapor in the air is unsaturated, that is, its partial pressure is lower than the vapor tension, which, for water at 20° C., is about 17 mm of mercury. If liquid water is put in contact with the open air it will therefore completely evaporate. If, however, it is put in the presence of the same atmosphere but in a closed vessel, as evaporation goes on the partial pressure of the water vapor will increase until it equals the vapor tension. After this, no further evaporation will take place unless the temperature is raised. The dependence of vapor tension on temperature may be strikingly shown by inserting a small quantity of water into a barometer tube. Fig. 149 shows a series of such tubes at temperatures 25° apart. In the tube at 0°, the mercury stands practically at its normal level, showing that the vapor tension of the water at that temperature is very low. At higher temperatures the mercury level is depressed. The difference P between the depressed and normal levels records the vapor tension of the water. A curve drawn through these levels gives a curve quite like that in Fig. 148.

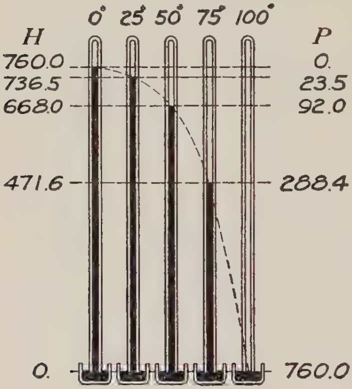


Fig. 149

237. Boiling.—The phenomenon of boiling differs from that of quiet evaporation in that vaporization takes place not only on the surface but also in the body of the liquid. Small bubbles form in the interior of the liquid. They expand as they rise into levels of lower hydrostatic pressure, vaporization continually going on at their boundaries. The pressure on these bubbles is equal to the total pressure of the gas above, plus the hydrostatic pressure corresponding to their depth below the surface. They will collapse if this pressure exceeds the vapor tension at that temperature. It is clear, therefore, that boiling cannot take place except when the total gas pressure on the liquid is less than the corresponding vapor tension, regardless of whether the partial vapor pressure is less or not. The vapor tension curve may also be called the boiling point curve, the pressures now signifying the total pressure on the liquid. If other gases were excluded, the situation would be much simpler. Boiling would then take place, instead of quiet evaporation, whenever the temperature of the liquid was raised above that which is characteristic of the existing pressure.

The change of boiling point with change of pressure is comparatively great, amounting for water near atmospheric pressure to about 0.37° C. for 1 cm of mercury.

238. Heat of Vaporization.—The heat of vaporization is defined as the amount of heat necessary to vaporize 1 gram of liquid without change of temperature. It varies considerably with the temperature and pressure under which vaporization takes place. For water under normal conditions, 100° C. and 760 mms pressure, the heat of vaporization is 538 calories per gram. As one increases the pressure and thus raises the boiling point, the heat of vaporization becomes rapidly less. The large value of the heat of vaporization of water makes it a useful material for many purposes. When steam condenses, considerable heat is liberated, a fact which is utilized in steam heating. When water evaporates a correspondingly large amount of heat is absorbed which results in a cooling of the surroundings. In fact, if water is caused to boil vigorously by a rapid reduction of pressure, it may easily be cooled until it freezes.

This may be seen in the following experiment. A small dish of water is placed under the bell jar of an air pump, and the air exhausted. As the pressure goes down it falls below the vapor tension and the water boils. This reduces the temperature. The pressure must then be reduced still further to maintain the boiling. To make the reduction of pressure sufficiently rapid, a small amount of sulphuric acid may be placed under the bell jar to absorb the water vapor. When the temperature reaches about 0° C., the water solidifies even in the form of bubbles which are emerging from the boiling liquid.

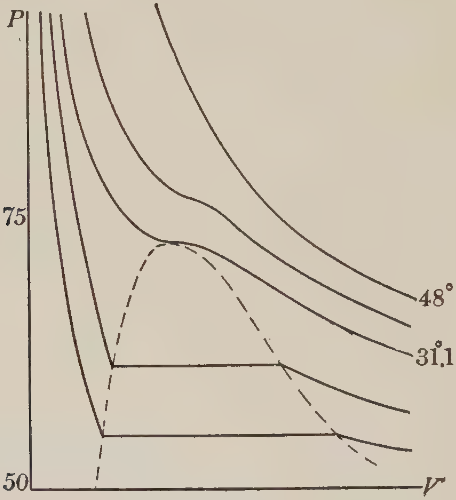


Fig. 150

239. Critical Point.—Fig. 150 shows the phenomenon of vaporization on a diagram whose coördinates are pressure and volume. The full lines are curves of constant temperature and are called isothermal lines. These curves and the values of the coördinates are given for carbon dioxide, although the curves are quite typical of this change of state for all materials. The dotted curve gives the position on the isothermals at which vaporization or condensation takes place. Within this region the material exists in both states at once and has values of P

and T which correspond to the points on the vapor tension curve. Within this region the isothermal curves are straight horizontal lines, indicating that the pressure does not change during vaporization. The lengths of these indicate the changes of volume during vaporization. It will be noted that these get shorter as the transition temperature rises, until finally a temperature is reached at which the change of volume during change of state becomes zero. This is called the critical temperature. This limiting point on the diagram at which transition from liquid to vapor just ceases taking place is called the critical point. There is a particular temperature, specific volume, and pressure for each substance which is characteristic of this phenomenon. For carbon dioxide it occurs at 31.1° C. and 73 atmospheres. The specific volume at the critical point is .0066 of its value for carbon dioxide gas at 0° C. and 1 atm. In other words, the density is about 150 times as great as under standard conditions.

This phenomenon may be easily seen if carbon dioxide is sealed into a glass tube whose volume is such that liquid carbon dioxide fills about two-thirds of the tube. At this point there is a well-defined surface of separation between the liquid and vapor. If the tube is heated, when the temperature reaches 31° the surface of separation will flutter and disappear. At this temperature the density of liquid and vapor become identical, and one is unable to distinguish between them. On cooling, the surface of separation may be made to reappear at the same point in the tube.

It was essential for the success of this experiment that exactly the right amount of material should be sealed into the tube. Since the total volume is fixed, its ratio to the mass of material enclosed must equal the critical volume which is defined in cc. per unit mass. The path on the pv diagram which represents this experiment will be a vertical line passing through the critical point, since v is constant. If too little mass had been taken, *i.e.*, too large a specific volume, the vertical path would not have passed through the critical point but would have emerged from the region of double state too far to the right. We would then have seen the liquid gradually boil away and the surface of separation sink to the bottom of the tube. If too much material had been taken, the path would have been too far to the left and would have emerged into the region of liquid state. In this case all the original vapor would have condensed, and the surface of separation would have risen to the top of the tube. The carbon dioxide would have become homogeneous in both cases, but at a lower temperature than the critical one. The surprising appearance at the critical point results from the maintenance of almost constant proportions of liquid and vapor up to the moment when they lose their identities.

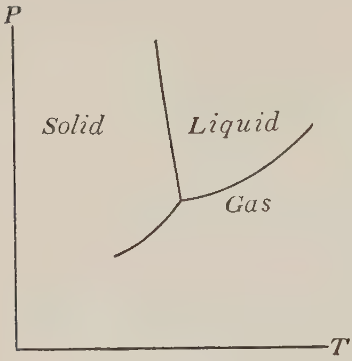


Fig. 151

It will be noticed in the diagram that isothermal lines for temperatures higher than the critical temperature do not pass through any transition region. In fact, if the temperature is raised high enough, the vapor takes on the character of a perfect gas. If one starts with vapor at a temperature lower than the critical, it is possible to liquefy by pressure alone. This is of great importance in the liquefaction of the so-called permanent gases. The critical temperature of nitrogen is −146° C. Only below this temperature can it be liquefied by compression. The critical temperature of water is about 374° C.

240. Sublimation.—The transition from the solid to the gaseous state is called *sublimation.* Ice vaporizes very considerably during the winter months. Other solids, such as camphor or iodine, waste rapidly by sublimation. A sublimation curve has the same general character as the melting or boiling point curves. Fig. 151 shows all three of these curves on one diagram. It will be noticed that the sublimation curve corresponds to pressures and temperatures lower than the other two. It is for this reason that in the case of water the phenomenon is not so well known.

241. Triple Point.—A most striking characteristic of Fig. 151 is that the three curves shown intersect in a point. That two of them, for example, melting point curve and boiling point curve, should intersect, signifies that at a certain temperature and pressure, solid and liquid may coexist, and also liquid and vapor. It may be shown by the principles of thermodynamics that at this point solid and gas must also be able to coexist, and therefore that the sublimation curve must pass through the intersection of the other two. This point is called the *triple point.* It is a point, and the only point, at which the three states may coexist. For water it occurs at + 0.0076° C. and 4.6 mms pressure. Ice below this pressure may be vaporized but not liquefied. The triple point of carbon dioxide lies at – 79° C. and 5.1 atm. pressure.

Liquid carbon dioxide can exist at room temperature, but only under a pressure of many atmospheres. Such conditions are met within the strong containers in which carbon dioxide is frequently stored. If the stopcock of such a container is opened and the pressure thereby reduced to one atmosphere, the rapid vaporization cools the material below – 79° C., its triple point, and solidification takes place. Solid carbon dioxide can exist at room pressure if the temperature is low enough. Its own sublimation keeps the local temperature down. In fact, the conduction of heat from the surroundings is usually so slow that the solid carbon dioxide wastes away very slowly.

PROBLEMS

1. An aluminum calorimeter weighing 80 grams contains 400 grams of water at temperature 30° C. A piece of dry ice is dropped into the calorimeter and the temperature of the water reaches 12° C after the ice is all melted. The calorimeter and contents now weigh 561 grams. Determine the heat of fusion of water. Specific heat of aluminum is 0.21.

2. A hole is drilled in a block of ice whose temperature is 0° C., and a block of metal weighing 120 grams and having a temperature of

98° C. is dropped into the hole. When the metal has cooled to the temperature of the ice, it is found that 16.2 grams of ice have been melted. What is the specific heat of the metal?

3. A copper calorimeter, sp. heat .09, weighs 80 grams. It contains 500 grams of water at 32° C. 120 grams of ice at 0° C. are put into it. What will be the temperature of the water when the ice is all melted?

4. In the preceding problem, if the amount of ice had been 220 grams, what would have happened?

CHAPTER XVIII

THERMODYNAMICS

242. The First Law of Thermodynamics.—Throughout the study of this subject we have implied that heat is a form of energy not fundamentally different from the other forms we have already encountered, and capable of being transformed into these other forms in accordance with definite laws. We have gone so far as to picture the actual state of the molecules in a body which contains "heat energy" and thus suggest a reduction of the problem back to mechanics. It is wise to recall that this point of view was not always maintained. Carnot, one of the ablest workers in the early history of this subject, held the view that "caloric," or heat energy as we now call it, was itself conserved; that it could be transferred from one body to another and even made "latent" but never could be destroyed or created. It was Robert Mayer, in 1842, who first clearly stated the principle of Conservation of Energy, including the first law of thermodynamics, which is its application to thermal processes.

This law may be stated as follows: "If a body be changed from one thermal state to another, the algebraic sum of the 'mechanical equivalents' of all the energy added to it or subtracted from it is the same, regardless of the way in which the change is brought about."

If, for example, water is to be changed from the liquid state at 20° to saturated vapor at 100°, the heat added, plus any work of compression and minus any work of expansion, is always equivalent to the same number of units of mechanical work, whether the water be vaporized at 20° and then heated as vapor or whether it be heated as a liquid to 100° and then vaporized. This is true, moreover, in spite of the differences of heats of vaporization at the two temperatures.

243. Mechanical Equivalent of Heat.—The first law of thermodynamics suggests at once the possibility of converting the calorie into an equivalent number of units of mechanical work. Experiments for this purpose have been performed with great care and precision, chiefly by Joule. He was able to raise the temperature of water by agitating it with a system of paddles and to measure very exactly both the rise in temperature and the mechanical work done. Later, Joule's work was repeated by Rowland in America, and since then many additional determinations have been made, also by electrical methods. The accepted result of this work is

$$1 \text{ calorie} = 4.19 \times 10^7 \text{ ergs} = 4.19 \text{ joules.}$$

The principle just stated has already been implied in many of the deductions made in these pages. For example, it has been said that part of the heat necessary to raise the temperature of a gas under changing volume is used to do the mechanical work of expansion. We believe that to be a quantitatively definite transformation, and we are now in a position to calculate what fraction of the heat applied is so transformed. The relation has been deduced for a perfect gas,

$$c_p - c_v = \frac{R}{M}.$$

Hydrogen has a molecular weight of 2. R has been found from the equation for perfect gases to be 8.3 joules per degree. Therefore, for hydrogen,

$$c_p - c_v = 4.15 \text{ joules per degree per gram,}$$

or very nearly 1 calorie per degree per gram. If $\frac{c_p}{c_v} = 1.4$ for hydrogen, one may easily compute

$$c_p = 3.5 \text{ calories per gram per degree.}$$
$$c_v = 2.5 \text{ calories per gram per degree.}$$

244. Isothermal and Adiabatic Expansion.—It has been shown in the study of Mechanics that the work done by a gas in expanding at constant pressure is equal to $p(v_2 - v_1)$. A curve showing the successive states of the gas plotted on a pv diagram would be a straight line parallel to the v-axis, and the work would be equal to the area below this curve. If the pressure is not constant during expansion, the work done is still given by the area below the expansion curve, as was shown in §106. It is easy to see that the work done in changing the volume of a gas from v_1 to v_2 depends on the way in which the pressure, and therefore the temperature, is allowed to change. A very interesting expansion is that in which the temperature is kept constant. The expansion curve for a perfect gas is then given by the equation $pv =$ cons. Such a curve was shown in §121. It is a rectangular hyperbola and is called an isothermal curve. A series of such curves may be drawn for different values of T in the gas equation $pv = RT$. They will all be tangent to the p- and v-axes at very distant points. Those corresponding to higher values of T will lie farther from the origin. Such a set of curves is very useful in practice, as it allows one to read off the temperature of a perfect gas for any values of p and v.

If a gas is allowed to expand along one of these curves its internal energy will remain constant, since this depends on the temperature alone. In order to keep the temperature constant, it is therefore necessary to furnish heat from outside equal to the work done by expansion. Both the work done and heat supplied are given by the area under the curve if it is plotted in suitable units. Between a given pair of volumes the work of expansion (or compression) will be greater, the higher the temperature. Calculation shows it to be proportional to the absolute temperature.

Another interesting case of gas expansion is that in which no heat is permitted to enter or leave the gas while its volume is changing. Such a process is called adiabatic. The work of expansion must now be done at the expense of the internal energy of the gas, and the temperature must fall. The curve on the pv diagram must therefore descend more sharply than the isotherms in order to reach the larger volume at a lower temperature. The amount which the temperature falls will depend on the nature of the gas as shown in its specific heats. The equation of this curve may be shown to be pv^{γ} = cons., where the exponent $\gamma = \frac{c_p}{c_v}$, and it is equal to 1.4 for most diatomic gases. The work done is again equal to the area under the curve. Since the work is done at the expense of the internal energy, it is easy to compute if the initial and final temperatures are known. It is, in fact, equal to the heat in ergs which must be supplied to produce this change of temperature when the volume of the gas is kept constant; that is,

work per gram of gas = $c_v(T_2 - T_1)$.

Fig. 152 shows a series of isothermal lines and one adiabatic line superimposed. It must be kept in mind that these represent the behavior of a *perfect* gas.

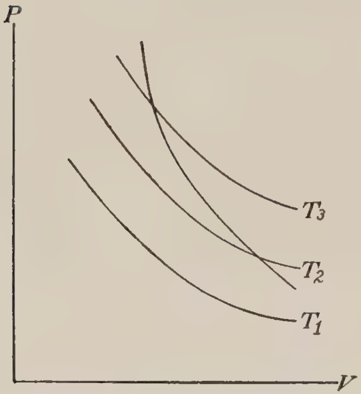


FIG. 152

245. Carnot Cycle.—If any heat engine is to be used as an agent for the continuous transformation of heat into mechanical work without replenishment of material, it is necessary that the material which undergoes compression and expansion should be periodically restored to its original state. Such a set of processes is called a cycle, and the material used is called the agent. At the close of a cycle it is evident that the agent has its original store of energy, and that any work done has been due to heat reservoirs in contact with the agent.

Fig. 153 shows a cycle which has great historical as well as present practical importance. It was first analyzed by Carnot. The cycle consists of two isothermal and two adiabatic curves. The figure is drawn for a perfect gas and may be thought of as follows: The gas is originally at the higher temperature T_2 and the highest pressure, as indicated by the point A. It is then allowed to expand isothermally to B, during which time it does work. This energy has been furnished by a heat reservoir (H) at this higher temperature T_2. The gas is now removed from contact with any outside heat source and the gas is allowed to expand further until its temperature falls to T_1 (C) adiabatically. It is then connected with a heat reservoir (C) at the lower temperature T_1 and compressed to D.

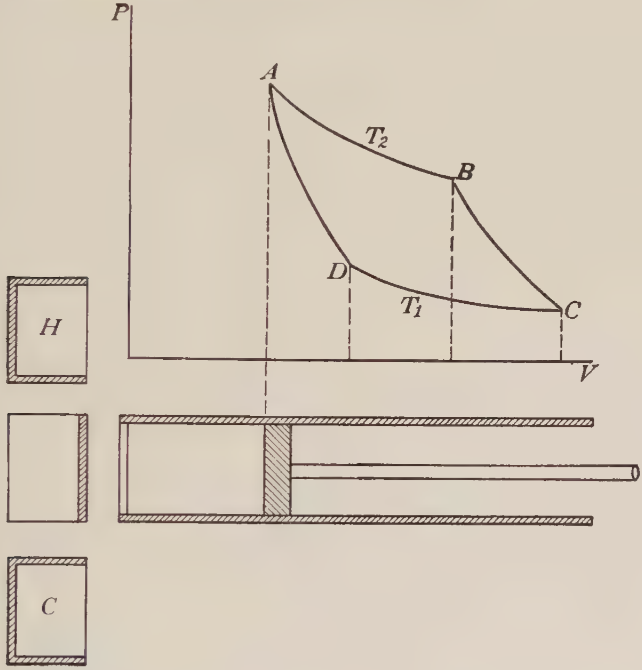


FIG. 153

During this compression, work is done on the gas and heat flows out to the low temperature reservoir. The agent is then again isolated and compressed adiabatically to A where its temperature is again T_2, and the agent is restored to its original condition. Four items of work have been done during the cycle, two positive and two negative. The work done during the adiabatic changes has exactly canceled, since both items are equal to C_v $(T_2 - T_1)$ and have opposite signs. All net effects are therefore due to the two isothermal parts. The work done isothermally may be calculated by integration, and it turns out to be proportional to the absolute temperature for a given ratio of end volumes. Moreover, the ratio of volumes at B and A may be shown equal to the ratio at C and D from the equations of the curves. Therefore we may write,

$$\frac{Q_2}{Q_1} = \frac{W_2}{W_1} = \frac{T_2}{T_1},$$

where Q_2 = heat flow during expansion at temperature T_2, and W_2 the work done during the same process. The total work done during the cycle is equal to $W_2 - W_1$ and is indicated by the enclosed area.

We may define the efficiency of the cycle as the ratio of net work done to the heat furnished at the higher temperature.

$$\text{Efficiency} = \frac{W_2 - W_1}{Q_2} = \frac{Q_2 - Q_1}{Q_2} = \frac{T_2 - T_1}{T_2}.$$

A cycle between the temperatures 400° and 300° abs. will have an efficiency of 25 per cent, *i.e.*, of the heat furnished at 400° only 25 per cent will be transformed into mechanical work; the remaining 75 per cent will flow out into the reservoir at 300°.

246. The Second Law of Thermodynamics.—The second law of thermodynamics differs fundamentally from the first in that it foretells the direction in which processes take place and thereby defines equilibrium conditions and the relations between them. The far-reaching consequences of this law are revealed only after careful mathematical analysis. In fact, it is not easy to make a complete statement of the law without using terms which we have not yet defined. The following partial statement is useful for our purposes.

"There are processes in nature which go forward spontaneously in one direction only." The second law asserts that these processes are *essentially* unidirectional, that after such a process has gone forward the bodies which took part cannot be restored to their original condition without the help of outside bodies. The original condition of the outside bodies cannot then be restored without further help, etc. Thus, after one of these irreversible processes the world as a whole has suffered a permanent change.

Processes which go forward spontaneously are all of this kind. For example, if two pieces of iron at different temperatures are brought in contact, heat will always flow from the hotter to the colder, never from the colder to the hotter. The Carnot cycle does not operate spontaneously, and it may be reversed. If reversed, it results in transporting heat from the lower temperature reservoir to the higher, but this must be accompanied by the performance of mechanical work by an outside agent, as a falling weight. If that weight is restored to its old level by the cycle running forward, the same amount of heat must be transported from higher to lower temperature. If any heat engine had a higher efficiency than the Carnot cycle when operating between the same temperatures, that is, could transform into work a larger percentage of the heat taken from the hotter reservoir, then it might be used to run the Carnot cycle backward and the net result would be a transfer of heat from low temperatures to higher, all other effects being canceled. This is forbidden by the second law, and we must conclude that the maximum efficiency which any heat engine may have is

$$\frac{T_2 - T_1}{T_2}.$$

This fact is so fundamental and free from characteristic properties of any substance that it has been used as a definition of temperature. We see, moreover, that if we use a thermometer whose expanding material is an absolutely perfect gas, its temperature will agree exactly with the thermodynamic scale.

247. Evaporation Cycle.—In order to apply the formula in the preceding section to a cyclic process, it is necessary for all the heat to be taken in at one temperature T_2 and given out at another T_1; that is, the cycle should be bounded by two isothermal and two adiabatic curves. When this is not the case, the efficiency is still the ratio of work done to heat furnished, but there are not two definite temperatures in terms of which this may be expressed. Nevertheless, it will always be true that if the cycle can be exactly reversed it is more efficient than a similar cycle which cannot be reversed. If heat is allowed to flow directly by conduction from a hot body to a cold one, no mechanical work is done. The process is irreversible. If such a process forms part of a cycle, the efficiency is bound to be lower than it would have been if conduction between different temperatures could have been prevented.

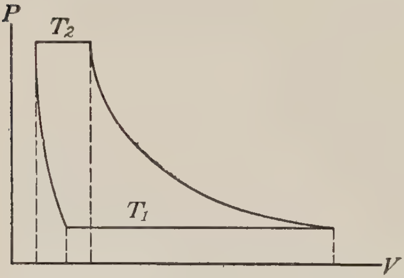


FIG. 154

Moreover, any cycle which is reversible and bounded by isothermal and adiabatic lines will have exactly the same efficiency as the Carnot cycle, that is, $\frac{T_2 - T_1}{T_2}$. For example, in Fig. 154 the isothermal lines are lines of constant pressure and represent a vaporization of the working substance, say water. The operation consists therefore of evaporation at constant temperature T_2, then an adiabatic expansion of the mixture of liquid and vapor, then partial condensation at temperature T_1, and finally adiabatic compression to the original liquid state. This is an idealization of the process in a steam engine. If the cycle is carried out reversibly its efficiency is $\frac{T_2 - T_1}{T_2}$. If unit mass is evaporated at T_2, the heat furnished is L, the heat of vaporization at that temperature. Therefore, since

$$\frac{W}{L} = \frac{T_2 - T_1}{T_2},$$

$$W = \frac{L(T_2 - T_1)}{T_2},$$

and should agree with the area within the curve which may be measured with a planimeter. If for practical reasons any process in the cycle is not reversible, then the efficiency is less, and for unit mass of water evaporated at T_2, less work will be obtained than that calculated above.

248. Simple Steam Engine.—Fig. 155 shows in principle the essential parts of a simple steam engine. Steam is admitted through the chamber S and is ejected through the opening e. The valve V slides back and forth over the openings into the two ends of the cylinder. For the position indicated in the figure, steam is entering the left end of the cylinder C, and is being ejected from the right end. The piston is moving to the right. At this moment the valve is moving to the left and will presently close p_1, the entrance to C.

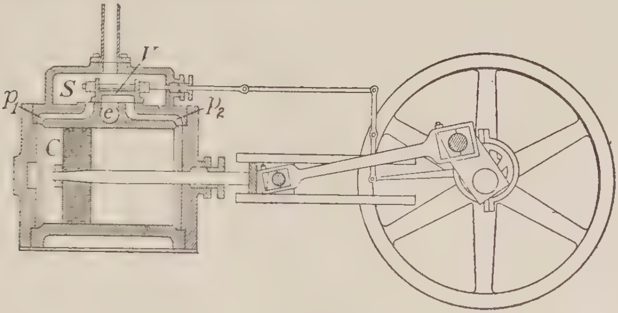


Fig. 155

Steam is admitted at a pressure of several atmospheres and a temperature correspondingly above 212° F. Motion of the piston during admission of the steam takes place at approximately constant pressure. After closing of the entrance port, the enclosed steam expands nearly adiabatically with falling temperature and pressure until C is connected with the exhaust e. The process is then continued by a return of the piston at the lower constant pressure, with an ejection of the steam until the exhaust port is closed. A cushion of steam always remains in the cylinder and further motion of the piston compresses this small quantity of material adiabatically until its pressure rises to that of the steam entering from the boiler.

This process is not strictly a cycle, since the greater part of the steam has not been restored to its original temperature and pressure. Fig. 156 shows an actual indicator diagram of this engine. Its enclosed area records correctly the work done during the process. The heat furnished must be calculated from that required to raise the temperature of water from that of the exhaust to that of the boiler and convert it into steam at the higher temperature. The efficiency is not given by the formula $\frac{T_2 - T_1}{T_2}$ except in rough approximation.

There are, moreover, many reasons why the practical operation of the engine has an efficiency less than ideal. During the entrance of the steam the pressure is not constant and is always less than that of the evaporating water in the boiler. A similar difference of pressure occurs at the exhaust. Moreover, since the expansion of the steam within the cylinder takes place with change cf temperature, the cylinder walls cannot be kept in thermal equilibrium with it, and heat will be lost by conduction. Far the greatest loss of efficiency takes place between the combustion of the coal and the boiler. The highest possible temperature which we might hope to use is that of the burning coal. The very considerable drop between that and the temperature of the entering steam represents a loss of efficiency far in excess of those losses which we have just been describing. This loss is largely avoided in the internal combustion engine. Here the heat is generated directly in the cylinder, and the expansion starts from the highest available temperature. The efficiency of the gas engine may be double that of the steam engine with boiler and coal combustion.

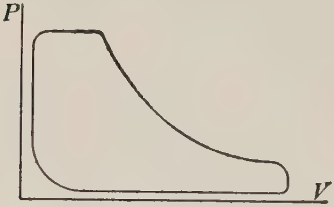


FIG. 156

249. Multiple Expansion Engines.—Since the efficiency of an engine has the maximum value $\frac{T_2 - T_1}{T_2}$ an attempt is made to increase this by increasing the temperature difference $T_2 - T_1$. To make T_1 as small as possible, the steam issuing from the cylinder is condensed and the temperature thereby reduced to that of the available water supply which is used for this purpose. Modern engines are all of this condensing type. Still more effective is the use of high temperatures with corresponding high pressures for the entering steam. If one made use of a large temperature difference with a simple engine, the chief difficulty would be the considerable loss due to difference of temperature between steam and cylinder walls. To avoid this, engines are constructed which divide the fall of pressure among a series of cylinders, each using the exhaust steam of the preceding for the forward stroke of its own piston. Fig. 157 shows an indicator diagram of a triple expansion engine which has divided the former cycle into three. It is possible in this way by packing and external heating to keep the cylinder walls always very near the temperature of the steam inside. This type has also important advantages of design.

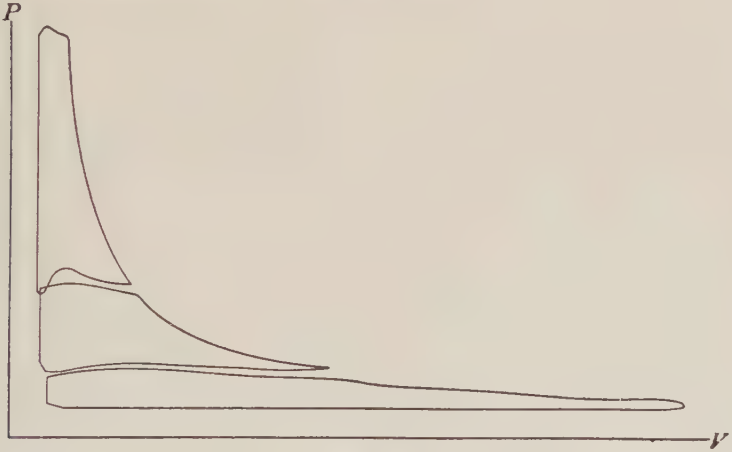


FIG. 157

250. Refrigeration.—If a Carnot cycle is run backward, the heat Q_1 is taken from the cooler reservoir where it was previously delivered. The larger amount Q_2 is now delivered at the hotter reservoir, and this is accomplished by doing an amount of mechanical work $Q_2 - Q_1$. This is the principle involved in the usual refrigerating machine. The temperature T_1 is usually so low that the vapor pressure of water is inconveniently small. A working material like ammonia is therefore employed. The efficiency of this process is defined as the heat drawn from the cold reservoir divided by the work done. That is,

$$\text{Efficiency} = \frac{Q_1}{W} = \frac{T_1}{T_2 - T_1}.$$

Contrary to the usual case, this expression may exceed 100 per cent. It will be noticed that a big temperature difference now reduces the efficiency. If, for example, T_2 is constant, say room temperature, and the temperature T_1 of the body being cooled goes down, the amount of work necessary to pump up a given amount of heat will rapidly increase. In the process of reaching extremely low temperatures, this is somewhat obviated by cooling in successive stages so that $T_2 - T_1$ does not become excessively large. Nevertheless, the efficiency will fall with T_1, even if $T_2 - T_1$ is kept constant. This is a serious difficulty in reaching the neighborhood of absolute zero.

251. Inefficiency of Ordinary Heating.—It has been pointed out that the thermodynamic inefficiency of heat conduction is one of the greatest practical losses. In the steam engine this has been mentioned as a most serious waste which engineers have found difficult to avoid. A striking example of this occurs in the familiar process of heating a room by combustion of coal. The temperature of the fire is greatly in excess of that of the room to which the heat is allowed to flow. The second law of thermodynamics states that such an irreversible flow of heat represents a corresponding loss of opportunity for transformation into mechanical work which can never be regained.

Lord Kelvin has suggested an ingenious device for heating a room based on the Carnot cycle as used in refrigeration. If the room to be heated is used as the hot reservoir and the air outside as the cool reservoir, an expenditure of mechanical work could result in pumping the heat from outside into the room. In fact, for moderate difference of temperature, the heat Q_2 delivered to the room might be many times as great as the mechanical work done. This is shown by the formula

$$\frac{Q_2}{W} = \frac{T_2}{T_2 - T_1}.$$

For $T_2 = 300°$, $T_1 = 270°$, this factor has the value 10. The room therefore receives not only the heat equivalent of the work done, but in addition nine times that amount of heat pumped in from out of doors where the temperature is 30° lower. If, therefore, the heat of combustion could be efficiently turned into mechanical work, we could make use of the high temperature which it involves to introduce into the room more heat than we now receive by conduction.

PROBLEMS

1. The burning of a pound of coal produces about 14,000 B.T.U's. If this energy could be used in lifting coal out of a mine 100 ft. deep, how many tons of coal could be taken out of the mine for every pound burned? Account for the fact that this result is about 20 times larger than that usually attained in practice.

2. The earth moves in its orbit nearly 19 miles per second, or about three million cms per sec. What is the heat equivalent of the energy of each gram of the earth's mass due to this motion? What would be the result if a body moving with such a velocity were suddenly stopped by collision with another body?

3. When the sun's rays are allowed to fall at right angles to a surface, it is found that 1.95 calories of heat are brought to each sq. cm of this surface per minute. What is the area of the surface if it receives energy at the rate of one kilowatt?

4. A train of mass 300 tons is hauled up a 2-per-cent grade at a speed of 30 miles per hour. 1,500 h.p. are used in doing this; find the total heat developed per minute as a result of friction.

ELECTRICITY

CHAPTER XIX

CHARGE, FIELD AND POTENTIAL

252. Electric Charges.—It is about 250 years since Robert Boyle published the first book devoted to the subject of electricity. The phenomenon which first formed the subject of study was electrification by friction. It had been known long before Boyle that if amber is rubbed, it becomes capable of attracting light objects. It is, in fact, the Greek word for amber (elektron) from which the word electricity is derived. Boyle already knew that the property of electrification must be extended to a large group of substances, and he was able to show not only that the electrified body was capable of attracting light objects but that two electrified bodies would attract or repel each other. If, for example, a glass rod is rubbed with a piece of silk and suspended by a silk cord, it will be repelled by another similarly rubbed glass rod. If, on the other hand, it is approached by a rod of sealing wax which has been rubbed with a piece of flannel, it will be attracted. This led very early to the conclusion that there were two kinds of electricity and that like kinds repel and unlike attract. Quite arbitrarily the electricity on the rubbed glass was called positive and that on the sealing wax, negative.

These phenomena may be clearly seen if one suspends a small pith ball by a dry silk thread. If an electrified body is brought near, the pith ball is attracted, regardless of the sign of the electricity. The previously uncharged pith ball behaves as if it had acquired a charge of sign opposite to that of the electrified body by being near it. This phenomenon is called induction, and will be discussed later in detail. If now the body touches the attracted pith ball, it appears to share its own charge with the pith ball, and the induced charge disappears.

The ball is then repelled by the like charge on the body. It is now charged by conduction, and its sign may be tested by approaching it with rubbed glass or sealing wax.

We are aware that electric charges may be produced by many means other than friction, as, for example, by the chemical action of a battery or the operation of a dynamo. But however the charge is produced or isolated, its properties are quite the same. These early experiments display therefore the operation of fundamental laws which underlie all other phenomena, however complicated.

253. Electric Fields.—Let us suppose we have before us a charged body. We say that the region surrounding the body is the residence of an electric field. Just as in the case of the earth which we believe to be surrounded by a gravitational field such that any mass brought near it is acted on by a force, so in the case of the charged body, any electric charge brought into its field will be acted on by a force. In the case of the gravitational field of the earth, we say that the force is everywhere directed toward the center of the earth. We might explore this field with a small mass, and with suitable apparatus for measuring force we could map it in both direction and magnitude. The customary way of doing this is by means of lines of force. These lines are drawn so that they have at every point the direction of the force on the test body at that point. Such lines in the earth's gravitational field would be radii pointed toward the center of the earth. The intensity of an electric field at a point is measured by the force which unit charge will experience if brought to that point. It may be plotted as one line per sq. cm for each unit of field intensity. A set of radial lines with uniform angular spacing would map the gravitational field in both direction and magnitude.

To determine the direction of the lines of force in an electric field, it is necessary to choose arbitrarily the sign of the test charge. This is selected once for all as positive; and we say therefore that lines of force emerge from positive charges, since the positive test charge is repelled by them, and end on negative charges, since the positive test charge is attracted by them.

254. Coulomb's Law.—Coulomb was able to show as early as 1785 that if two charged bodies are brought near each other, the force of attraction or repulsion varies as the inverse square of the distance between them. He made use of a torsion balance to measure the forces and he found that the inverse square law held more rigorously, the smaller the charged bodies were in comparison to the distance between them. He showed also that the force was proportional to the product of the quantities of electricity on the two bodies.

With the help of this result we may now define a unit of electric quantity as follows: *A unit quantity of electricity will repel an equal and like quantity at a distance of 1 cm with a force of 1 dyne.* This requires that the two charges in question shall be distant from other bodies which contain charges or on which charges may be induced. This includes the air itself. The measurement must be made *in vacuo*.

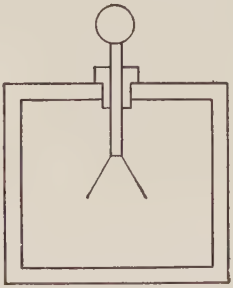


FIG. 158

The results of Coulomb's experiment may then be stated in the equation

$$F = \frac{qq'}{r^2},$$

where q and q' are the quantities on the two small bodies and r the distance between them.

The unit of electricity which has been defined thus is called the electrostatic c.g.s. unit or, more briefly, the electrostatic unit (*esu*). For many purposes this unit is too small to be convenient. The coulomb is the "practical" unit of quantity and is equal to 3×10^9 esu. It will be made clear later why a unit of this magnitude has been chosen.

255. The Electroscope.—The electroscope is an instrument which may be used to detect and measure electric charges. A common form is shown in Fig. 158. It consists of a glass vessel, through the opening of which is introduced a metal rod. This rod has at its upper end a metal knob and at the lower end a metal stirrup over which hangs a metal foil. The metal rod is held in the mouth of the vessel by a material through which electricity does not easily pass, such as hard rubber or amber. Such a material is called an insulator. In contrast to this, the metal of the rod allows the electricity to pass very easily. All metals have this property, and are called conductors.

When the electroscope contains no charge the two "leaves" hang vertically. If a charge is given to the metal ball at the top it will spread immediately over the rod and leaves. The leaves now have charges of like sign and will repel each other. The greater the charge on the electroscope, the farther apart they will stand. It is possible to calibrate the instrument and read off the magnitude of the charge from the deflection of the leaves.

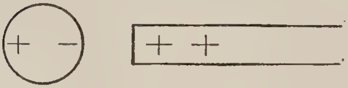


Fig. 159

256. Induction.—It is clear that although the electroscope may indicate the size of the charge on it, it does not directly reveal its sign. This may be found most easily with the help of the phenomenon called induction. Modern research has convinced us that matter is made up of positive and negative charges, normally in equal quantities. To say that a body is uncharged or neutral does not imply that no electricity is resident in it. It signifies only that there is no surplus of the one over the other. If now a neutral body is brought into an electric field, the individual charges will be acted upon by forces which tend to separate the positive from the negative. When a glass rod (positive charge) approaches a neutral pith ball, the negative charges of the ball will be drawn toward the side nearer the glass and the positive will be in excess on the opposite side. The pith ball as a whole will then be attracted toward the glass. This distribution of charge is shown in Fig. 159.

Similarly with the electroscope. If a rubbed glass rod is brought near the metal knob but not in contact, the leaves will diverge. The knob has become negatively charged, and the leaves positively. This effect becomes greater the nearer the rod is brought to the metal knob. If, then, the electroscope had been previously charged with, say, positive electricity, the approach of the glass rod would cause the leaves to diverge still farther. If, however, the electroscope was originally charged negatively, the effect might be more complicated. While the rod is quite distant, the small induced positive charge will partially neutralize the original negative and the leaves will begin to collapse. As the rod is brought nearer they will collapse still farther, and if the original charge is not too great, it may finally be overwhelmed so that the leaves begin to diverge again, now with the induced positive charge. If the rod is now removed without having been in contact with the electroscope, the original state is restored. By this means the sign of the electroscope charge may be determined without altering its magnitude.

257. Charging by Induction.—We have already seen how a second body may be charged by actual contact with a previously charged body. This is called charging by conduction. In this case electricity actually passes from one body to the other. When charges are induced on the electroscope by the approach of a charged body, the positive and negative charges on knob and leaves are equal in amount and will neutralize each other again if the charged body is removed. If, however, one connects the knob to the earth by touching it with the finger, the leaves will collapse even if the charged body is still present. By making the electroscope part of a larger conducting body it has been possible for the repelled induced charge to recede still farther and leave the leaves neutral. They remain collapsed, even though the contact with the earth is now broken. The attracted charge on the knob was unaffected by earthing; and if the charging body is now removed, the charge on the knob will be released and spread over the system, diverging the leaves once more. The electroscope is now charged with electricity of sign opposite to that of the inducing body.

258. The Atom.—It has long been believed by chemists that all matter is made up of definite individual parts which they have called atoms. The word atom means something indivisible, and indicates the point of view held by the early workers in this field. The properties of elements and compounds are clearly due to the peculiarities of these atoms and their combinations. What their structure actually is has eluded scientists until very recently. Even now comparatively little is known, but we have strong evidence of some outstanding facts.

Two very celebrated physicists, Rutherford and Bohr, have contributed greatly to this problem. They have not only added to the evidence, but to a still greater degree they have organized and interpreted our information.

The present accepted view is that every atom consists of a nucleus of positive charge about which revolve smaller negative charges, called electrons, much in the way in which the planets of our solar system revolve about the sun.

Almost the entire mass of the atom is attributed to the nucleus. This mass, as well as the amount of the positive charge, varies from element to element. The electrons, however, are quite the same in all atoms. We believe, in fact, that they may be exchanged between atoms without any changes of properties. The negative charge of the electron is a definite and fixed quantity—the smallest amount of electricity which can be isolated. The magnitudes of all nuclear charges and, therefore, all charges, are multiples of it. It is the atom of negative electricity.

A neutral atom is one which has just enough electron planets to neutralize the positive charge of its nucleus. This number is called the atomic number, and varies from element to element. For the most part, it increases with the mass of the atom, but not proportionally. Thus neutral hydrogen has one electron, helium 2, nitrogen 7, sodium 11, gold 79.

When an atom or combination of atoms in a solution or gas has too few or too many electrons, it is said to be ionized, and is called a positive or negative ion. In solids the equilibrium positions of the nuclei are fixed and form the definite crystalline structure of the material. It is probably true in the case of metals that the electrons may move about in this nuclear structure with comparative freedom, always, however, in obedience to the attractions and repulsions of the nuclei and other electrons. If some of the electrons are drawn away, the body is positively charged; if they are present in excess, it is negatively charged.

259. Radioactivity.—In 1896, Becquerel, a French physicist, published the first account of what we now call Radioactivity. The striking phenomenon observed with radioactive materials was that they emitted corpuscles or electromagnetic waves, or both, which were detected by their power to ionize gases, to cause scintillations in a properly prepared screen, or to blacken a photographic plate. These rays have for the most part the power of penetrating considerable thicknesses of material. Perhaps the most remarkable property of radioactive materials is that, as their activity continues, the original substance is gradually transformed into a new element with new atomic weight and new chemical properties.

260. Radioactive Radiations.—Early investigations of the radioactive radiations disclosed the fact that they could be separated into three types, which were called alpha, beta and gamma rays. The alpha rays were shown to consist of positively charged particles. This may be revealed by their deflection in an electric field. A beam of alpha rays may be allowed to pass through a small aperture and produce a bright spot on a fluorescent screen. An electric field at right angles to their path deflects the spot in the direction of the field. The beta rays, on the other hand, are a stream of negatively charged particles and are deflected in the opposite sense by the same electric field. For a given position of the screen it may easily be shown that the deflection of the spot depends on the velocity of the particles, their charge and their mass. Later experiments have shown that beta rays all have the same charge and are, in fact, electrons moving with great velocity.

The alpha particles are much heavier. They experience very slight deflections even in intense fields. We know now that they are the nuclei of helium atoms with charges twice as great as the electrons but with a mass about 7,000 times as great. Their mean velocity is, nevertheless, about one-tenth that of the beta particles and about one-fifteenth of the velocity of light. The gamma rays differ from both the others in that they experience no deflection whatever in the most intense fields. Moreover, they are capable of penetrating matter in much greater thicknesses than either alpha or beta rays. Their penetrating power is, in fact, considerably greater than that of X rays, with which they were early identified. There is no doubt now that they are electromagnetic waves identical in nature with X rays and ordinary light, except that their wave length is extremely short. The wave length of most gamma rays is shorter than that of any other radiation, the origin of which we know.

261. Radioactive Disintegration.—The first substance which was discovered by Becquerel to give out these radiations was uranium, the element of greatest atomic mass. Mme. Curie soon after discovered that thorium possessed the same property. A by-product of this process was a gas, thorium emanation, also radioactive. The discovery of radium soon followed.

The interesting result of these discoveries and the others which came after them was that it was finally possible to show that during the emission of alpha and beta rays, the parent material disintegrated and became a new element. The radioactive elements were finally arranged in two series starting with uranium and thorium, both heavy atoms with large nuclear charges, and proceeding through a sequence of elements into which they transform successively until they all finally become lead, an element which is not observably radioactive.

This disintegration consists of a breaking down of the nucleus, the characteristic part of the atom. The alpha and beta particles are ejected from it during its activity. We must believe therefore that the heavier nuclei are themselves complex structures made up of positive and negative charges in unequal amounts. The loss of a beta particle (electron) increases the positive charge by one unit, but leaves the mass practically unchanged. The loss of an alpha particle (helium nucleus) reduces the positive charge by two units, and the mass by the mass of the helium atom. It must be remembered that it is the nuclear charge and therefore the number of electronic planets which determines the chemical properties.

262. Paths of Alpha Particles.—The penetrating power of the alpha, beta and gamma rays now became the subject of careful study. Bragg discovered that the stopping power of different materials increased with the atomic weight. Rutherford allowed a beam of alpha particles to penetrate a foil of solid material and studied the scattering out of the original narrow beam. He discovered the important fact that the number of particles deflected through more than 90°, and consequently returned toward the source, was surprisingly large. This could only be accounted for on the basis that the atoms contained a highly concentrated nucleus and that if the alpha particle, in passing through the atom, passed close to this nucleus, it came under the influence of an intense electric field and was widely deflected, as a comet might be in our solar system. It is not difficult to see that we cannot expect by Coulomb's law to obtain an intense field from a given charge unless the radius is small. The field is greater the nearer the point is to the center, so long as it is not actually within the charge.

More recently these experiments have been repeated by C. T. R. Wilson and others in such a way that the details of this phenomenon may be seen much more clearly; the earlier work is thus completely confirmed. If a chamber of water vapor is supersaturated by sudden expansion and alpha particles are passed through it, those molecules which are ionized by the alpha particles become centers of condensation. The alpha particle thus leaves behind itself a track of water droplets which may be seen and photographed. The sharp deflections of the alpha particles are then very apparent, and leave no doubt that the nucleus has an extremely small radius, probably of the order of 10^{-13} cm. The radius of the atom is about 10^{-8} cm. The ratio of the radius of the nucleus of the atom to the radius of the outer orbit of its electrons is therefore of the same order of magnitude as the ratio of the sun's radius to the mean distance of its outer planet. The atom must be thought of as empty in the same sense as our solar system is empty. Fig. 160 is a diagram showing paths of alpha particles and the deflections which they suffer near nuclei or electrons. It must be remembered that the condition favorable for a wide deflection of the alpha particles is not simply that it encounter a region of intense field, but that the center of that field have a large mass and therefore remain nearly stationary during the collision. Thus the electrons do not cause such wide deflections.

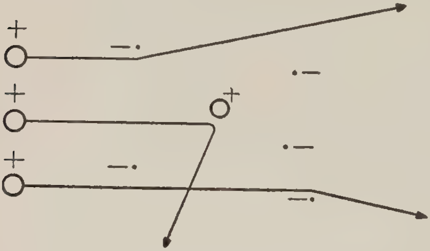


FIG. 160

263. Methods of Ionization.—It has been pointed out that whenever a molecule or atom loses or gains an electron it is said to be ionized. The first experiments described indicated that this could be accomplished by friction. The details of this operation are intricate and not altogether understood. It is doubtless related to other phenomena brought about by contact of dissimilar materials, which we shall discuss later.

We know that a gas consists of molecules in continual motion, and that these molecules frequently suffer collisions or, from our present standpoint, frequently come into regions of intense electric fields due to their neighbors. If these collisions are violent enough, they result in the transfer or the tearing loose of electrons which a little later recombine to neutralize the molecules once more. This thermal ionization is a very small effect for air under ordinary conditions. It is much greater for metal vapors and, in the atmospheres of the sun and stars, rises to considerable proportions.

Ionization by high speed electrons or, more particularly, by alpha particles is a more important phenomenon. The energy of these particles is great because of their large masses and velocities. An electroscope may easily be discharged by passing alpha particles through the air near it. The ionization of the air provides both positive and negative charges, and the electroscope attracts to itself a sufficient number to neutralize it. This is, in fact, a method for measuring radioactive effects.

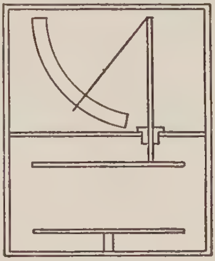


FIG. 161

Fig. 161 shows an electroscope designed for this purpose. Charge is introduced on the upper plate, and the consequent deflection of the leaves may be measured on a scale. Radioactive material is placed on the lower platform, and the rate of discharge of the electroscope noted.

Still another method of ionizing gases consists in passing through them radiation of short wave length, such as X rays or gamma rays. Energy necessary to remove the electrons is absorbed from the radiant beam and may be reëmitted if the electron and ion reunite. Such penetrating radiation is always present unless a region is carefully shielded. It is partly due to radioactive materials all about us and to some extent it may originate outside the earth. Due to this cause the air is always slightly ionized.

264. Measurement of Electronic Charge.—Several measurements of the charge of an electron have been made. The most precise of these was carried out by Professor Millikan at the University of Chicago, by means of his celebrated oil drop method. Fig. 162 shows the essential features of his apparatus. Two plates are given charges of opposite sign by connecting them to the terminals of a battery. The air between them is ionized by a beam of X rays from the X-ray tube at the right.

A fine spray of oil is introduced through the tube A, and small drops find their way down through a hole in the upper charged plate. A drop soon picks up an electron, or perhaps two. It is then pulled upward by the charges on the plates. These charges may be varied until the upward force exactly equals the downward pull of gravity. A telescope on the left is focused on the drop so that one may make this precise adjustment of the electric field. Then we may set

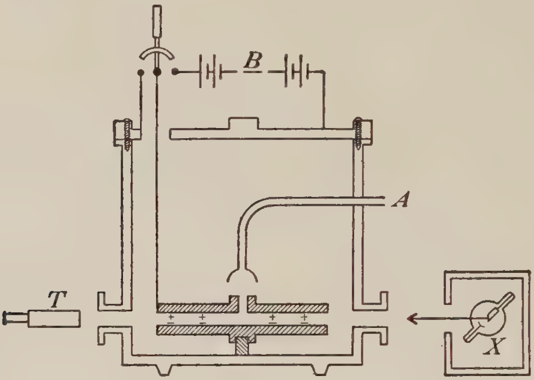


FIG. 162

$$mg = Eq,$$

where m is the mass of the drop and g the acceleration due to gravity, E the electric field intensity in dynes per unit charge, and q the magnitude of the charge. E may be calculated from the electric circuit and distance between the plates; m is measured by allowing the uncharged drop to fall through air and measuring its velocity. Very small particles do not fall through air with a constant acceleration, as is the case for large bodies. They fall with a nearly constant velocity which depends on their size. This relation has been determined theoretically and allows one to measure the small mass. q measured this way was found to be always once, twice or occasionally three times a small constant value, depending on whether the drop picked up one or more electrons. Measurements extended over several years always gave these small multiples of the elementary unit.

The value which is now accepted as the most precise for the electronic charge is 4.774×10^{-10} esu, or 1.591×10^{-19} coulomb.

265. Potential.—We have already learned that in the vicinity of a small charged body there exists an electric field whose direction and magnitude may be explored and determined by means of a positive unit test charge, and that the field may be mapped by means of lines of force. Moreover, if the body is small compared with the distance from it, the intensity of field is given by the equation

$$E = \frac{q}{r^2},$$

where q is the charge on the small body and r is its distance from the point in question. Unit charge therefore is acted on by a force equal to $\frac{q}{r^2}$, and this force is directed radially outward or inward according to the sign of q. If the positive test charge is moved against the force due to the field, work will have to be done upon it, just as work must be done on a mass in order to move it against the force of gravity. If the body is positively charged, it will repel the test charge, and work must be done in bringing the test charge into the field from a point outside it. This work may be computed by multiplying the displacement by the component of the force in its direction; but since in this case the force changes as we proceed, the computation requires the help of the calculus. If the force is $\frac{q}{r^2}$ one may show that the work necessary to bring unit charge from an infinite distance to a point a cms distant from the charged body is $\frac{q}{a}$. To take unit charge from a point whose distance is a to one whose distance is b requires $\frac{q}{b} - \frac{q}{a}$ units of work. It must not be forgotten that these expressions for field and potential apply to the case of a point charge, that is, a charged body whose dimensions are small in comparison with the distance to the point at which the potential or field is being computed. They must not, for example, be used for the region between two charged plates, as in the Millikan experiment.

The potential at any point in an electric field is defined as the work necessary to bring unit positive charge from an infinite distance to the point in question. Its unit in the electrostatic c.g.s. system is an erg per esu of charge. In the practical system it is a joule per coulomb. This practical unit is called a *volt*. Two points have a difference of potential of one volt if it requires one joule of work to carry a coulomb of electricity from one to the other. It may be easily shown that 1 esu of potential = 300 volts.

The idea of potential is analogous to that of level in the gravitational field. It would be quite possible to measure altitudes not, as is customary, in feet above sea level, but rather in terms of the work necessary to transport unit mass from some standard point to the point in question against the force of gravity. This work is always equal to mgh, and if $m = 1$ and g is known, this work may be used to measure h.

We say in the gravitational problem that masses always tend to move toward lower levels, they seek a minimum potential. So in electric fields, the positive charge always tends to move toward points of lower potential and, if no other forces are present, arrives at the second point with kinetic energy equal to the work necessary to put it back or, in other words, equal to the difference of potential multiplied by the magnitude of charge. Similarly, negative charges move toward points of higher potential.

266. Equipotential Surfaces.—Equipotential surfaces are made up of points which have the same potential. They correspond to constant level lines on a contour map. Since difference of potential is a measure of the work done in carrying a charge from one point to another, it follows that a charge may be moved in any direction on an equipotential surface without work being done. Therefore the electric field must lie at right angles to these surfaces at every point. If it did not it would have a component parallel to the surface, and work would be done in moving against it.

We have learned that the potential due to a point charge q is given by the expression $\frac{q}{r}$, where r is the distance from the charge. Lines of force are here radial lines, and the equipotential surfaces are spheres with the charge as center.

267. Superposition of Fields.—When two or more charges are present, the field at any point may be found by adding the fields at the point due to the charges separately. This addition must be made vectorally, as was the case with forces. Potential, like work, is not a vector but a scalar. The resultant potential is found by adding the contributions from the separate charges algebraically. As an example, suppose two equal positive charges are fixed at a given distance from each other, say $2a$, and we wish to find the field intensity and potential midway between them. The field intensity due to the charges separately will be equal in magnitude and opposite in direction; the resultant is therefore zero. This means that if a positive test charge is placed at this point it will be repelled equally by the two charges, and the resultant force acting on it will be zero. The potential at this point, however, is $\frac{q}{a}$, due to each charge and the sum of these two equal scalar quantities is $\frac{2q}{a}$. This means that although no force operates on the test charge at this point, work must nevertheless be done to bring it there. If the two equal charges had been opposite in sign, the potential at the midpoint would have been zero and the field intensity,

$$\frac{2q}{a^2}.$$

268. Conductors.—When a conducting body is brought into an electric field, electricity will flow in obedience to that field as long as it is applied or until the charges have reached a distribution which just neutralizes the applied field. A conducting material must clearly be one in which the electrons may move about from atom to atom with comparative ease. This is the case with metals, whether the atomic nuclei form a regular crystalline structure or whether they have the more random arrangement of a liquid. Ionized gases and liquids are also conductors, since they contain electrons and ions which are comparatively free to move.

Substances which do not have this property are called non-conductors. Their atoms retain the electrons with considerable permanence. These atoms may be somewhat distorted or polarized by electric fields, but they furnish no free electrons or ions. This is predominantly the case for non-ionized gases, pure water and many other liquids and solids.

If a conducting body is placed in an electric field, its electrons will move into such positions that, in spite of the outside field, all points of the conductor have the same potential. Only then can it be in equilibrium. Just as a liquid in a container will flow under the forces of gravity until the surface has a single level, so the electrons in a metal body will move until the field throughout the metal has disappeared and the potential has a single value.

By the potential of a conductor in equilibrium is meant the potential of any one of its points. In some cases it is possible to compute this directly. For example, a charged metal sphere out of the influence of other bodies will have a uniform charge over its surface and none in the interior. This follows from the mutual repulsion of the charges and the symmetry of the body. To calculate the potential, we observe that, since the charge is uniformly distributed over the surface of the sphere, the field outside the sphere will be symmetrical and the lines of force will be radial. The field is, then, the same as if the entire charge were at the center of the sphere. Since the potential at a point in such a field is q/a, we may decrease a until it becomes equal to the radius r. Therefore the potential at the surface of the sphere is q/r. This potential is shared by all interior points as well and is called the potential of the sphere.

A volt is equal to 1/300 esu of potential. It will require 1/300 esu of charge to give a sphere of 1 cm radius a potential of 1 volt. Since 1 esu of charge $= \frac{1}{3 \times 10^9}$ coulomb, $\frac{1}{300}$ esu of charge $= \frac{1}{9 \times 10^{11}}$ coulomb. Therefore $\frac{1}{9 \times 10^{11}}$ coulomb will give a sphere of 1 cm radius a potential of 1 volt.

Since the solid metal sphere is free from charge in its interior, all of the above computations will still be valid if the sphere is hollow or has cavities of any number or shape. A metal container acts as an electric screen, protecting the interior from all outside fields. Moreover, all metal bodies will, when in equilibrium, have their charges resident on the outer surface, though not usually uniform in distribution. In general, bodies have the greatest density of charge where the curvature is greatest. This accounts for the great charge density at conducting points. The result of this is a high intensity of field just outside and a consequent strong tendency to discharge through the ionized air.

269. Capacitance.—The capacitance of a body is numerically equal to the charge required to give it unit potential. For isolated bodies the potential is always proportional to the charge. Therefore the capacitance, $C = \frac{q}{V}$, depends only on the size and shape of the body and, as we shall see later, on the medium which surrounds it.

For an isolated sphere, $V = \frac{q}{a}$; therefore $C = \frac{q}{V} = a$. The capacitance of an isolated sphere in free space is equal to its radius. In other words, the capacitance of a sphere of 1 cm radius is 1 esu of capacitance. In the practical system, the unit of capacitance is the farad. A body has a capacitance of

1 farad if a charge of 1 coulomb gives it a potential of 1 volt. If a sphere of 1 cm radius has a charge of $\frac{1}{9 \times 10^{11}}$ coulomb, its potential is 1 volt. Its capacitance is therefore $\frac{1}{9 \times 10^{11}}$ farad. Thus, 1 esu of capacitance = $\frac{1}{9 \times 10^{11}}$ farad. Since the farad is so large a unit, the microfarad is commonly used. 1 farad = 10^6 microfarads. 1 microfarad = 9×10^5 esu of capacitance.

The earth may for most purposes be thought of as a conducting sphere. Its capacitance is very large; consequently moderate charges do not appreciably change its potential. It is very convenient therefore to use it as a fixed standard of "zero" potential. Any conductor which is metallically connected to the earth shares its potential.

270. Measurement of Potential.—The electroscope, if provided with a scale, may be used to measure the potentials of charged bodies. Suppose the electroscope originally uncharged and then connected by a metal wire to the charged body, which is kept at such a distance that the induction between it and the electroscope is negligible. Electricity will flow until the potentials of the charged body and the electroscope become equal. If now the capacitance of the electroscope is small, as is usually the case, it will not rob the charged body of much electricity in order to attain its potential, and the original value will therefore not be appreciably changed. The deflection of the leaves indicates the amount of charge, and this is proportional to the potential. The scale may be calibrated directly in volts or any other convenient units.

271. Effect of Neighboring Bodies. Condensers.—Suppose a metal sphere has a positive charge. We know this charge is uniformly distributed over the surface and that the potential of the sphere is $\frac{q}{a}$, and that this may be verified with a suitably calibrated electroscope. If now a second sphere which is connected by a fine wire to the earth is brought near, a negative charge will be induced on it. This approaching negative charge reduces the potential of the original sphere. If the electroscope is still connected to it, the leaves will be seen to collapse as the second sphere approaches. It is no longer correct to say that the potential of the sphere is $\frac{q}{a}$. That expression is valid only when the sphere is alone. If the sphere is brought near to an earthed sphere, its potential approaches zero and its capacitance increases. It would be necessary now to increase its charge in order to regain the old potential. In this case the difference of potential of the two spheres is equal to the potential of the first, since the second is grounded. Such a pair of conductors separated by a non-conductor is called a condenser. The charge on a condenser is the charge on one of the bodies or "plates." Condensers are usually designed so that practically all the lines of force from one conductor end on the other, and the positive and negative charges are therefore equal. The difference of potential of a condenser is the difference of potential of its two plates. Its capacitance is its charge divided by its difference of potential. This is equal to the capacitance of one of its conductors if the other is earthed. Condensers with large capacitances will therefore hold large charges at small differences of potential.

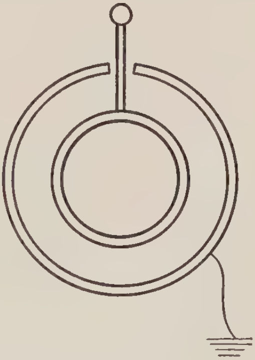


FIG. 163

272. Capacitance of Condensers.—In simple cases, the capacitance of a condenser may be computed directly from its design. Suppose, for example, that the two plates consist of a pair of concentric spherical surfaces, as shown in Fig. 163.

Since the spheres are concentric, the field between the two surfaces is symmetrical. It is exactly the same as if the inner sphere were alone in space. The potential difference between two points in this field is $\frac{q}{a} - \frac{q}{b}$. If a is the smaller radius and b is the radius of the outside sphere, $\frac{q}{a} - \frac{q}{b}$ becomes the potential difference between the two spheres.

The capacitance

$$C = \frac{q}{V} = \frac{1}{\frac{1}{a} - \frac{1}{b}} = \frac{ab}{b - a}.$$

If $b - a$ is small the capacitance is large. If $b - a$ is very small and therefore a almost equal to b, we may set $b - a = t$, and $ab = a^2$.

$$C = \frac{a^2}{t},$$

where t is the thickness of the dielectric, or

$$C = \frac{4\pi a^2}{4\pi t} = \frac{A}{4\pi t},$$

where $A = 4\pi a^2$, the area of one of the plates. If t is given in cms and A in sq. cms, C will be the capacitance in esu. Since 1 microfarad is equal to 9×10^5 esu, it will be necessary to divide this result by 9×10^5 to convert it to microfarads.

273. Dielectrics.—Faraday discovered in his experimental researches that if the insulating medium in a condenser is altered, the capacitance of the condenser is changed. The calculations just made were based on the supposition that the space between the plates was free from matter. If a non-conducting material is introduced, the capacitance will be multiplied by a factor which is characteristic of the material. This constant factor is called the "dielectric constant," and will be denoted by the symbol k. For air k is only slightly greater than unity and may be set equal to one for most purposes. For glass, $k = 3$, for paraffin, 2, for mica, 6–8.

Faraday's explanation of this phenomenon was that the new medium weakened the field between the plates, so that for the same charges on the plates, the work necessary to carry electricity across and consequently the difference of potential was reduced. We now say the same thing in other words. We believe that a dielectric differs from a conductor in that no appreciable number of electrons is free to move about or is made free to move about by the application of an electric field. The field does, however, polarize the molecules or atoms to some extent by pulling their positive and negative charges in opposite directions. The effect of uniformly polarizing a group of atoms in one direction will be neutralized by neighbors in the interior but will result in a "bound" positive charge on one boundary and a bound negative charge on the other. Thus, if a block of paraffin is placed between two charged plates a bound negative charge will occur near the positive plate and a bound positive charge near the negative plate. This is shown in Fig. 164, in which the plates A and B are the charged conductors and the circles indicate the polarized molecules. These bound charges do not enter into the value of q in the expression for C, since that indicates the charge put on the plates from outside. It does, however, alter the value of potential difference which is due to all the charges actually in the condenser.

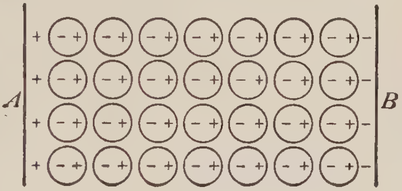


FIG. 164

For the spherical condenser we may therefore write more generally,

$$C = \frac{kA}{4\pi t}.$$

274. Plane Condensers.—So long as the distance between the plates of a condenser is small compared with their area, we may say quite generally that the capacitance per unit area is $\frac{k}{4\pi t}$. This is particularly useful as an approximation for plane condensers, a very common type made up of a large number of alternate plates of metal and dielectric. Alternate metal sheets are connected together and form respectively the positive and negative plates of the condenser, as in Fig. 165. It will be noted that both sides of the inner metal sheets are adjacent to the dielectric and must enter the expression for area. Here again,

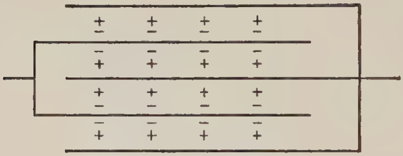


FIG. 165

$$C = \frac{kA}{4\pi t},$$

where A is the effective area of positive or negative plate and t is the thickness of dielectric.

275. Combinations of Condensers.—If a group of condensers is connected, as shown in Fig. 166, so that all the positive plates together form one larger plate, and similarly for the negative, the condensers are said to be connected in parallel. The capacitance of the combination is then the sum of the separate capacitances.

$$C = C_1 + C_2 + C_3 + \text{etc.}$$

If the condensers are all alike, this follows directly from the formula in the previous paragraph. In the more general case, it may be noted that the difference of potential V is the same for each condenser. The charges may be different. Calling them q_1, q_2 and q_3, the total charge is

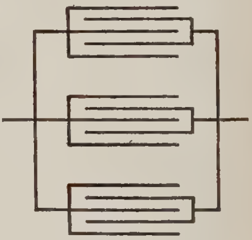


FIG. 166

$$q = q_1 + q_2 + q_3.$$

Total capacitance,

$$C = \frac{q}{V} = \frac{q_1 + q_2 + q_3}{V},$$

$$q_1 = C_1V, \text{ etc.},$$

so

$$C = C_1 + C_2 + C_3.$$

Condensers connected as in Fig. 167 are said to be connected in series. In this case, if a charge q_1 is placed on a plate of the first condenser, equal charges will be induced throughout the series. Thus the positive charge at A will equal the negative charge at B, which again equals the positive at C and so forth. The total external charge placed on the combination is therefore equal to the charge on each condenser, or

$$q_1 = q_2 = q_3 = q.$$

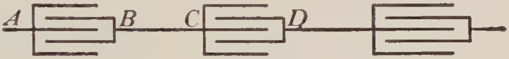


FIG. 167

The potential difference will vary with the capacitance, since $q = C_1V_1$, and the total difference of potential

$$V = V_1 + V_2 + V_3.$$

So

$$\mathrm{C} = \frac{q}{V} = \frac{q}{V_1 + V_2 + V_3} = \frac{q}{\frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}},$$

from which $$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$[1]

Connecting condensers in series therefore reduces the capacitance below the least single capacitance of the group.

276. Energy of Charged Body.—When a body is charged, work is required for the process. To bring up any small element of charge requires an amount of work equal to the amount of charge multiplied by the potential of the body being charged. Each additional element of charge increases the potential and makes it more difficult to add the next increment of charge. The case is analogous to the work done in laying a brick wall. The first row of brick requires a minimum amount of work.

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[1] In deriving this formula, the capacitance to earth of the exterior surface is neglected. In most cases the error is insignificant. See *Science* May 9, 1930, p. 488.

Each succeeding row must be raised to a level proportional to the number of rows already laid. Since the work is proportional to the height, the total work may be set equal to the total weight of the bricks, times half the final height to which they are raised. So in this case, the potential of the body is proportional to the charge already there, and the total work may be written

$$W = \tfrac{1}{2}\, qV.$$

We may think of this as a type of potential energy, just as in the case of the brick wall.

Since

$$C = \frac{q}{V},$$

$$W = \tfrac{1}{2}\, qV = \tfrac{1}{2}\ CV^2 = \frac{1}{2}\cdot\frac{q^2}{C}.$$

The energy of a condenser is found by the same reasoning and is given by the same expression, always remembering that q is the charge on one plate and V the difference of potential of the plates.

PROBLEMS

1. Two small pith balls each weighing .1 gram are suspended from the same point by silk threads 10 cm long and of negligible weight. They are then equally electrified. They repel each other so that each thread makes an angle of 45° with the vertical. What is the charge on each ball?

2. Three insulated conducting spheres are placed far apart and then charged till their potentials are 3, 5, and 7 esu respectively. If their radii are 4, 5, and 6 cm respectively, find the potential of the whole system when they are connected by a very fine wire.

3. Two large plane metallic plates are placed parallel to each other and 1.6 cm apart. They are charged to a potential difference of 48 ergs per unit charge. Find the intensity of electric field at a point between the plates and express it in volts per cm.

N.B. The field is uniform throughout; why?

4. In Millikan's oil drop experiment, if a drop has a charge equal to that of a single electron, find the force acting upon the drop if the potential difference between the plates is 3,000 volts and the distance between the plates is .9 cm.

5. Two conducting spheres, one having a diameter four times that of the other, are connected by a long thin wire and charged. Compare their charges, their potentials, and their energies.

6. A conducting sphere 16 cm in diameter is charged with 400 esu of electricity. What is the energy of the charge? It is now made to touch another equal sphere previously uncharged; then the two are separated by a long distance. Find the total energy of the charge. The charge is the same in the two cases; why should the energies be different?

7. Find the capacitance of the earth in farads, taking its radius as 63×10^7 cm.

8. Find the work done when a condenser of 8 microfarads is charged to a potential of 220 volts. If this condenser is now made to share its charge with another condenser of 4 microfarads, find the potential difference between the terminals of each condenser and the energy of the entire charge. Why is this total energy less than the original energy? What has become of the difference?

9. Suppose all the minute particles of water in a cloud are of the same size and that they all carry equal charges. Suppose, further, that they are far enough apart to be independent of each other. If 1,000,000 such particles were to unite to form a single spherical rain drop, how would the capacitance of the drop compare with the total capacitance of the particles before they united? How does the potential of the charge compare with that before the particles united? Does this suggest any reason why lightning frequently accompanies rainfall?

10. Three condensers having capacitances of 2, 3, and 5 microfarads respectively are connected in series, and 220 volts applied to the outside terminals. Make a sketch. Find the charge in each condenser and the potential difference between the terminals of each condenser.

11. A condenser is composed of two square plates 20 cms on a side. The plates are placed parallel to each other and one mm apart. Find the capacitance of the condenser. If 500 ergs are used in charging it, what is the potential difference between the plates and the charge on each plate?

CHAPTER XX

CURRENT ELECTRICITY

277. Conduction.—In the study of electrical phenomena thus far, we have been chiefly interested in charges accumulated on isolated bodies. We have been primarily concerned with static electricity, and yet it has been apparent that charges may travel from one body to another with great rapidity, provided there is a difference of potential between the two bodies, and provided further that a conducting path is furnished for the charge to follow. We are accustomed to think of metallic wires as furnishing such a conducting path. While the electric charge is in motion along the conductor, the conductor is said to carry an electric current. The term current has a quantitative significance and means the quantity of electricity conveyed per unit time. It is usually represented by the letter I. If Q units of electricity pass through a wire in t units of time, the average current is given by the equation

$$I = Q/t.$$

This should be considered as the defining equation for electric current. A unit current is one which conveys electric charge at the rate of one unit per second.

278. The Coulomb and the Ampere.—In §254 it is stated that in the practical system the unit of quantity of electricity is the coulomb, and that the coulomb is equal to 3×10^9 esu. This serves at present as a sufficiently accurate definition; but later, when we study the electromagnetic system of units, we find that all the units of the practical system are related by definition to those of the electromagnetic system.

The *ampere* is the unit of electric current in the practical system and, in accordance with the defining equation given in the preceding topic, it is the unvarying current which conveys one coulómb per second.

279. The Mechanism of Conduction.—In metallic conductors, it appears that the electrons associated with the atoms have a greater freedom to wander from atom to atom than have the electrons in the atoms of non-conducting substances. When a potential difference is established between different parts of a conductor, there is a drift of electrons toward places of higher potential; and since the electron is a negative charge, this is equivalent to the flow of positive electricity toward the places of lower potential. The individual electrons may move slowly, but the fact that they are set in motion all along the conductor at almost the same instant gives the impression of a very high speed. If a horizontal pipe several miles long were full of water and a little more water were pumped in at one end of the pipe, an equal amount of water would flow out of the other end at almost the same instant, and yet we should not say that the speed of the water was necessarily great. Likewise, if electrons flow in at one end of an Atlantic cable, others will flow out at the other end with only an insignificant time interval between the two phenomena.

280. Electrolytic Conduction.—The term electrolyte usually refers to a liquid solution which acts as a conductor of electricity. Water solutions of acids, bases, and salts are fairly good conductors of electricity, and are classed as electrolytes. When one of these substances is dissolved in water, it is partially ionized; *i.e.*, in the case of acids, the hydrogen in some of the molecules is dissociated from the acid radical, carrying with it a positive charge and leaving the radical negatively charged. Each hydrogen ion shows a positive charge equal in magnitude to that of the electron. The radical carries a negative charge consisting of as many electrons as there are hydrogen atoms united with it before the dissociation. The charged atom or the charged radical is called an ion. If two plates of metal are dipped into the solution and a sufficient difference of potential is established between them, the positively charged ions drift toward the negatively charged plate and the negative ions drift in the opposite direction; thus there are two sets of carriers for the current instead of one, as in the case of metallic conduction. When the hydrogen ions reach the negative plate, each receives an electron from the plate and becomes a neutral atom. A union of these atoms takes place to form molecules, and when their number is sufficiently great, bubbles of hydrogen appear on the plate. At the positive plate the negative ions give up their charges and react with the metal of the plate to form a salt, or with water to reform the acid and liberate oxygen. Some special cases of electrolysis will be discussed more in detail in another paragraph.

281. The Simple Voltaic Cell.—In the preceding paragraph we assumed two plates of metal in an acid solution to be maintained at a difference of potential by some agent outside the electrolytic cell. Suppose now that the two plates are of different metals—as, for example, copper and zinc—and that they are merely connected by a wire. If the solution is dilute sulphuric acid, we shall find that the evolution of hydrogen gas at the copper plate and the production of zinc sulphate at the zinc plate go on without any outside agent to maintain a potential difference. An instrument for measuring potential difference would, if connected to the two plates, show a potential difference between them. The wire, if tested by effects which we shall soon study, would be found to be carrying an electric current. It is to be observed that the hydrogen is liberated at the copper plate while the zinc is entering the solution at the opposite end of the cell. Ordinarily, when zinc is dissolved in dilute sulphuric acid, the hydrogen is liberated at the surface of the zinc where the chemical reaction takes place. The difference becomes clear when we consider that the current is carried through the solution by moving ions. The initial discovery in connection with these phenomena is due to Volta, who published an account of his work in the year 1800. The cell in which the phenomena may be observed is called a voltaic cell, and several such cells when connected together constitute a primary battery. The word primary used in this connection signifies that the battery is the source of electrical energy, and distinguishes it from a secondary battery, which is used to store energy supplied to it from some external source. In the simple voltaic cell the energy of chemical reaction between the zinc and the acid is the source from which the electrical energy is obtained. That there should be established a potential difference between the plates is to be expected since, when zinc goes into solution in the form of ions, each ion carries with it a positive charge, thus leaving the zinc plate with a corresponding negative charge. The zinc is, accordingly, called the negative plate of the cell, and the copper the positive.

282. Magnetic Effect of the Current.—While a treatment of the subject of magnetism and a further discussion of the relation of electric currents to magnetism will be reserved till later, it is desirable to mention here a most striking phenomenon that was discovered by Oersted in 1820. If, as in Fig. 168, a long copper wire is placed above a poised magnetic needle and parallel to it, no effect upon the needle is observed; but if the ends of the wire are now connected to the plates of a voltaic cell, the needle is deflected from its initial direction which was approximately north and south. It is neither attracted nor repelled by the wire, but remains in a horizontal plane, simply swinging around the vertical axis.

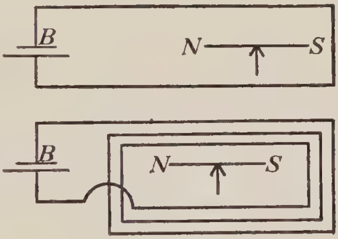


Fig. 168

When the wire is bent to form a loop and so placed that a part of the wire is above the needle and another part below it, the forces acting upon the needle are increased. This suggests that we might wind the wire around the needle many times and still further increase the effect. A circular scale may be placed under the needle so that its deflection can be measured. Such a piece of apparatus is called a needle galvanometer and is used to detect the presence of an electric current in a circuit. When properly made and calibrated, it becomes a measuring device for the current flowing through it.

Since there is found to be a torque existing between the coil and the magnet, it is immaterial which one of the two is allowed to turn. In the d'Arsonval galvanometer, the coil is suspended between the poles of a strong magnet; and when the current passes through the wire, the coil is turned through an angle that depends upon the strength of the current. Fig. 169 illustrates the construction of the d'Arsonval galvanometer. The wire is wound on a light frame and the frame is suspended by a fine wire between the poles of the magnet N and S. m is a small mirror which reflects a beam of light onto a scale. As the coil turns when a current flows through it, the spot of light moves across the scale. It is not our purpose to describe at this time the various instruments used in electrical measurements, but merely to point out that we have a simple and satisfactory means of ascertaining when a wire is carrying an electric current.

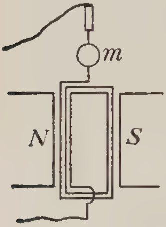


Fig. 169

283. Electrolysis of Metallic Salts.—The simplest case of electrolysis is that in which the electrolyte is a neutral solution of a metallic salt and the electrodes are of the same metal as that in the solution. For example, two plates of pure silver may be immersed in a solution of silver nitrate in water. The plate at which the current enters the solution is called the anode and the plate at which it leaves the cell is the cathode. After the current has been flowing for some time, it is found that the anode has lost in weight, that the cathode has gained in weight by the same amount, and that the average concentration of the solution remains unchanged. Every atom of silver going into the solution becomes a positive ion and carries a positive charge equivalent in magnitude to the charge of the electron. As many atoms leave the solution at the cathode in a given time as enter it at the anode. An atom of silver has a definite mass and an electron has a definite electric charge; hence there is an exact relationship between the charge passing through the cell and the mass of silver deposited. It is found experimentally that one coulomb is conveyed through the solution by .001118 gram of silver.

284. The Volt.—The difference of potential between two points has been defined as the work per unit charge done in carrying the charge from one point to the other. Expressed in the form of an equation,

$$V_1 - V_2 = \frac{W}{Q}.$$

In the practical system of units, W is measured in joules and Q in coulombs, and the unit is called a volt. A volt is equal to the potential difference between two points when one joule of work is done in carrying one coulomb from one point to the other. This definition of the volt has been given before.

285. Electromotive Force.—In any circuit, the work done in carrying a unit quantity of electricity completely around the circuit is a measure of the resultant electromotive force in that circuit. Since potential difference is also defined as work per unit quantity of electricity, it is evident that electromotive force and potential difference have the same physical nature and are measured in terms of the same units. The name electromotive force is unfortunate, since it is not a force. As a charge flows around a circuit, it usually flows from higher potential to lower potential in some parts of the circuit; then there must be other parts of the circuit where it flows from lower to higher potentials in order that it may return to the initial potential at the starting point. In a circuit consisting of a simple voltaic cell with plates connected by a wire, the places at which the current flows from lower to higher potential are at the surfaces of the plates. It is here that the chemical action takes place that furnishes the energy to produce the current. We think of the electromotive force in this simple case as localized in those places where energy in some other form is converted into electrical energy. If in a more complicated circuit there are places where electrical energy is converted into some form other than heat, it is customary to say that at such places there is a counter electromotive force. A running motor, while converting electrical energy into mechanical energy, produces an electromotive force in a direction opposing the flow of the current. An electrolytic cell in which electrical energy is converted into energy of chemical separation exerts a counter electromotive force.

286. Electrolysis with Platinum Electrodes.—We have already discussed in §280 the electrolysis of an acid solution and pointed out that hydrogen is liberated at the negative electrode. At the positive electrode the negative ions give up their charges; and if the electrode is insoluble in the acid, as is the case with platinum, there is a reaction with the water in the cell to liberate oxygen and form the acid. Thus, if sulphuric acid is used, we have

$$SO_4 + H_2O \rightarrow H_2SO_4 + O.$$

Aside from the reduction of the concentration of the solution near the cathode and the corresponding increase of concentration near the anode, the net result of the process is merely the electrolysis of water. Hydrogen and oxygen in the right proportions to form water are liberated, the hydrogen at one electrode and the oxygen at the other.

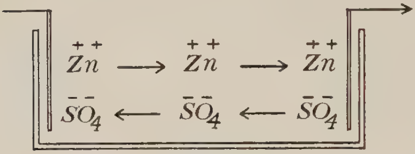

$\overset{++}{Zn}$ → $\overset{++}{Zn}$ → $\overset{++}{Zn}$
$\overset{--}{SO_4}$ ← $\overset{--}{SO_4}$ ← $\overset{--}{SO_4}$

Fig. 170

When a neutral salt solution, instead of an acid solution, is used with platinum electrodes, the result is different in these respects: metallic ions instead of hydrogen carry the positive charges, and usually the negative electrode is coated with metal instead of with hydrogen. Furthermore, the concentration of the salt solution diminishes and the acid content of the solution gradually increases. Fig. 170 may help to visualize the process.

287. Electrolysis with Soluble Electrodes.—In the preceding paragraph we assumed that the electrodes were of platinum and did not enter into chemical union with the acid. If the electrodes were composed of some metal that is easily soluble in the acid, no free acid would be formed, but the metal of the anode would go into solution as the cathode becomes plated with the metal that is already in the solution. The electrolysis of silver nitrate with silver electrodes, which has already been discussed, is an illustration. Further illustrations that make the process visible are worthy of mention. If we use a colorless salt solution—for example, zinc sulphate in water—and copper electrodes, the fact that the copper goes into the solution is apparent since the characteristic blue color of the copper salts appears at the anode almost immediately after the current starts to flow. A more intense blue color is obtained if an alkaline solution of ammonium nitrate is used, in which case a double salt of ammonia and copper is formed.

The electrolysis of lead acetate furnishes another excellent experimental illustration that may be projected on the screen. When the current passes through the solution, the lead is deposited in crystalline form. The crystals grow as the process continues and often assume beautiful geometric forms. When the current is reversed, the crystals that have already been formed go back into the solution and a new crystalline structure is built up at the other electrode.

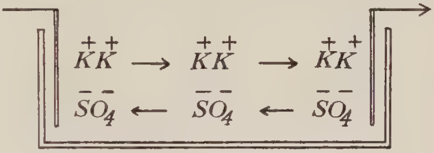


Fig. 171

288. Electrolysis of Salts of the Alkali Metals.—The fact that potassium and sodium react with water to form the hydroxides of these metals makes the electrolysis of their salts somewhat different from that of other metals. To illustrate, consider the electrolysis of potassium sulphate in water, using platinum electrodes, as in Fig. 171. The potassium ions go with the current to the right, and at the cathode, after giving up their charges, react with water as follows:

$$2K + 2H_2O \rightarrow 2KOH + H_2.$$

The SO_4 ions go to the anode and there enter into the following reaction:

$$2SO_4 + 2H_2O \rightarrow 2H_2SO_4 + O_2.$$

The result is, first, the formation of the acid and the alkali which, if brought together, would reform the salt potassium sulphate; second, the production of hydrogen and oxygen in the right proportions to form water.

289. Polarization.—If plates of two different metals are placed in an electrolyte, there is always a difference of potential between them; and when they are connected by a wire a current is produced in the wire. If the plates were made of the same metal, there is no reason for supposing that there would be a difference of potential between them, and none is observed unless there is a difference in temperature or in concentration of the electrolyte in the neighborhood of the plates. Starting with two platinum plates which when immersed in an acid solution show no difference in potential, suppose we pass a current through the cell from one plate to the other for a few seconds. We know that one of the plates will now be coated with a thin film of hydrogen and the other with a film of oxygen. The plates thus coated are no longer alike and they behave as if made of different metals, *i.e.*, there is a difference of potential between them. In this case it amounts to two volts. The cell is said to be polarized, and the two volts are called the electromotive force due to polarization. The direction of this electromotive force is such as to be in opposition to the current that produced the polarization. If we undertake the electrolysis of dilute sulphuric acid, using a cell to produce the current that has an EMF less than two volts, a current is found to flow momentarily, but only long enough to form the hydrogen and oxygen films; then the current decreases to zero, and all electrolytic action ceases. This is evidently because the counter EMF of polarization becomes equal to the applied EMF. To carry on electrolysis in this case a battery must be used having an EMF greater than two volts.

The following experiment furnishes the basis for the foregoing discussion. A long wire PQ, in Fig. 172, carries a current from P to Q, supplied by a storage battery. V is a voltmeter for reading the potential difference between *m* and *n*. *C* represents an electrolytic cell containing dilute sulphuric acid and having platinum electrodes. The potential difference which is read from the voltmeter is also that which is applied to the cell; and if a current flows through the cell, it will be indicated by the ammeter *A*. The contacts at *n* may be moved along the wire so that the potential difference between *m* and *n* may be varied. As we move *n* from *m* toward *Q* the ammeter shows no current until the potential difference as indicated by the voltmeter becomes two volts, then the current begins to flow and increases as the potential difference is increased.

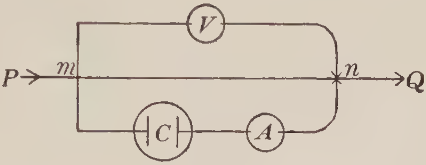


Fig. 172

290. Electrolysis without Polarization.—If the experiment just described is repeated with a cell in which the products of electrolysis do not alter the chemical nature of the exposed surfaces of the plates, there will be no polarization, and hence no counter EMF. In such a case, when the contacts *n* are moved from *m* toward *Q*, the ammeter begins to show current as soon as the voltmeter shows any potential difference. When copper electrodes are used in a copper sulphate solution, copper is removed from one plate and deposited upon the other, and neither plate is chemically different because of the operation. This process is used commercially in purifying copper. A large slab of the metal mixed with the impurities that it has brought with it from the blast furnace is used as the anode in an electrolytic cell, and at the cathode there is deposited electrolytic copper which is very nearly pure. Since there is little or no counter EMF, the process is carried on with small expenditure of electric energy.

291. Faraday's Laws of Electrolysis.—Nearly a hundred years ago, Faraday first announced the fundamental laws of electrolytic action as follows:

(1) The mass of an electrolyte decomposed by an electric current is directly proportional to the quantity of electricity passing through it.

(2) If the same quantity of electricity passes through different electrolytes, the masses of the substances liberated at the electrodes are proportional to their chemical equivalents.

These laws are both involved in the statement made in §280, that each monovalent ion in electrolysis carries a charge equal in magnitude to that of the electron, while the divalent and trivalent ions carry twice and three times as much respectively.

292. The Electrochemical Equivalent.—The electrochemical equivalent of a substance is defined as the mass per coulomb liberated in electrolysis. In §283 it was stated that the electrochemical equivalent of silver is .001118 grams per coulomb. Starting with this figure, we may compute the electrochemical equivalent of any other element by resorting to the proportion suggested in Faraday's second law. Suppose that we wish to find the electrochemical equivalent of zinc. The atomic mass of zinc is 65 and its valence is 2; therefore its chemical equivalent is 32.5. Let x represent the mass per coulomb, *i.e.*, its electrochemical equivalent. The chemical equivalent of silver is 107.88 and the mass per coulomb deposited is .001118 grams. Faraday's law states that these masses are proportional to the chemical equivalents; hence

$$107.88 : 32.5 = .001118 : x,$$

from which $x = .000337$ grams per coulomb. Since the chemical equivalents of most of the elements are known, it is a simple matter to find their electrochemical equivalents, and hence to find the mass deposited in electrolysis by any current flowing for a given time.

293. The Gram Equivalent.—The gram equivalent of a substance is a number of grams equal to the number representing the chemical equivalent. Thus 23 grams of sodium, 39 grams of potassium, and 35.5 grams of chlorine are the gram equivalents of these substances. Faraday's second law virtually says that the gram equivalents of all ions transport the same quantity of electricity. This quantity of electricity is easily computed. The gram equivalent of silver is 107.88 grams. One coulomb is conveyed by .001118 gram; therefore the number of coulombs conveyed by 107.88 grams is 107.88 ÷ .001118 = 96,500 coulombs. Problems such as those suggested in the preceding paragraph are also easily solved by making use of this number.

Problem.—Find the mass of copper deposited from a copper sulphate solution by 10 amperes flowing for 30 minutes.

Atomic mass of copper is 63.4.

Gram equivalent of copper is 31.7 grams.

10 amperes in 30 minutes carry $10 \times 1{,}800 = 18{,}000$ coulombs.

$$96{,}500 : 18{,}000 = 31.7 : x, \text{ from which } x = 5.91 \text{ grams.}$$

294. Charge per Ion.—It has been repeatedly mentioned that the charge carried by the univalent ion in electrolysis has the same magnitude as the charge of the electron. Probably no one has read this statement without raising the question in his own mind as to how we know it is true. In our study of mechanics of fluids, we learned that there are about 2.70×10^{19} molecules in a cc. of gas under standard conditions of temperature and pressure. The density of hydrogen is .0000898 gram per cc. If one gram of hydrogen carries 96,500 coulombs in electrolysis, .0000898 gram will carry $.0000898 \times 96{,}500 = 8.666$ coulombs. As there are two atoms per molecule in gaseous hydrogen, we should divide 8.666 by $2 \times 2.70 \times 10^{19}$ to obtain the charge carried by one hydrogen ion. This result is 1.60×10^{-19} coulomb. One coulomb = 3×10^9 electrostatic units. 1.60×10^{-19} coulombs = $1.60 \times 10^{-19} \times 3 \times 10^9 = 4.80 \times 10^{-10}$ electrostatic units. Millikan's result for the electronic charge, as explained in §264, is 4.77×10^{-10} electrostatic units. The difference between these two results is not greater than our uncertainty as to the figure 2.70×10^{19} for the number of molecules per cc. of the gas.

295. Polarization of the Voltaic Cell.—In the normal operation of the simple voltaic cell having zinc and copper electrodes, hydrogen is deposited on the copper plate and the cell becomes polarized in a short time after the circuit is closed. The presence of the hydrogen cuts down the EMF of the cell and thus reduces the current it can produce. On this account the cell is seldom used in its simple form in making primary batteries. Since hydrogen is responsible for the trouble, the remedy that suggests itself is to oxidize the hydrogen as fast as it is produced. In the dry cell that is used so extensively, manganese dioxide ($Mn0_2$) is used as a depolarizer, and in some cases mercuric chloride is added to the ammonium chloride solution that is used as the electrolyte. Some of the oxygen in the manganese dioxide is used in oxidizing the hydrogen, and a lower oxide of manganese is produced. The mercuric chloride reacts with hydrogen as follows:

$$HgCl_2 + H \rightarrow HgCl + HCl.$$

Zinc and carbon are used for the electrodes in the dry cell, and because of their large size the cell is capable of furnishing a large current. Ammonium chloride reacts with the zinc only while the current is flowing; hence the cell is well adapted for work in which there are long periods of inactivity. The name dry cell is somewhat misleading, since when the cell really becomes dry it is useless.

296. The Daniell Cell.—This cell is one of several in which two electrolytes are used. In this case the two are separated by a porous cup. The electrodes are of zinc and copper, the zinc being placed in the porous cup and surrounded with a zinc sulphate solution, while the copper is in the glass jar outside the cup and surrounded with a copper sulphate solution. The solutions fill the pores of the cup so that the current flows through its walls without difficulty.

In Fig. 173 the dotted lines represent the porous cup, the zinc sulphate is represented in an ionized condition at the left of the partition, and the copper sulphate at the right. When the circuit is closed, the positive ions move toward the right, zinc goes into solution and copper comes out on the copper plate. There is practically no polarization, since copper is deposited upon copper. As the cell continues to furnish current, the concentration of the zinc sulphate increases and that of the copper sulphate decreases. This must be compensated by occasionally throwing out some of the zinc sulphate solution and diluting the remainder with water. Fresh crystals of copper sulphate are usually kept around the copper plate; these dissolve as they are needed to replenish the supply of copper ions. The cell should be kept on a closed circuit most of the time to keep the metallic ions drifting toward the copper plate; otherwise the copper sulphate will diffuse through the porous cup and react with the zinc, coating it with a deposit of copper. In the gravity cell the difference in specific gravity is depended upon to keep the solutions separate, and no porous cup is used. Formerly many thousands of these cells were used in operating telegraph lines.

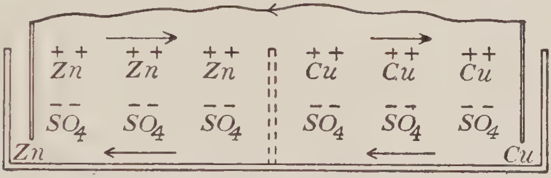


FIG. 173

297. The Standard Cell.—The fact that the Daniell cell does not polarize and that its EMF is therefore approximately constant, led, many years ago, to the suggestion that it might be used to furnish a standard EMF. At the present time a cadmium cell, known as the Weston standard cell, is the accepted standard of EMF. It is a two-fluid cell like the Daniell cell, and therefore does not polarize; and it has the advantage over other cells that have been used as standards in that its temperature coefficient is very small. The glass container consists of two tubes connected by a cross tube, as shown in Fig. 174. A platinum wire is sealed through the glass at the bottom of each tube to make the connection to the electrodes. One of the tubes contains pure mercury as the positive electrode, and the other contains a cadmium amalgam, which constitutes the negative electrode. A paste of mercurous sulphate covers the mercury, and crystals of cadmium sulphate are placed in both arms of the container. A saturated solution of cadmium sulphate fills the tubes, which are sealed to prevent changes by evaporation. When such a cell has been carefully made with pure chemicals, it has an EMF given by the following formula:

$$E = 1.0183 - .0000406(t - 20) \text{ volts.}$$

t is the temperature on the centigrade scale. When one of these cells is sent to the Bureau of Standards for calibration, a certificate is furnished which gives the EMF to six significant figures. The EMF will remain constant for a year or more within a limit of .01 of one per cent, provided it has proper care. The cell should never be made to deliver more than .0001 ampere, its temperature should be kept nearly constant, and it should not be subjected to mechanical shock when in use. It should be recalibrated as often as once in two years.

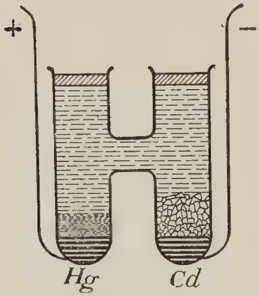


Fig. 174

298. The Storage Battery.—As is generally known, a storage battery is one whose energy may be replenished by passing a current through it in a direction opposite to that of the current which the battery delivers when in use. The reactions of the Daniell cell would be reversed if current from some external source were sent through it from the copper to the zinc electrode. Copper would go into solution and zinc would accumulate on the zinc plate. This cell is not to be recommended as a storage cell, but it illustrates very simply the reversibility of the chemical reactions involved.

The cell that is so widely used has dilute sulphuric acid as an electrolyte, and its electrodes are made of lead in various states of oxidation, depending upon the condition of the cell. A simple lecture-table experiment illustrates the storage of energy in the lead cell. Two plates of lead cut from the same piece and put into dilute sulphuric acid show no appreciable difference of potential and furnish little or no current when connected to an electric bell. If a current from another battery is passed through the cell for a few seconds, it will now be able to ring the bell for a few seconds; and if the current from the battery traverses the cell for a longer time, the ringing of the bell is prolonged. The current from the battery causes the electrolysis of the acid, depositing hydrogen on one plate and oxygen on the other. The hydrogen escapes if the lead surface is free from oxide, and the oxygen unites with the lead to form lead peroxide (PbO_2). When the cell is fully charged there is a large accumulation of this oxide, and the plates show a difference of potential of about two volts.

The chemical reactions in this cell are complicated and varied. They are different at different stages of the charge and discharge, and are dependent upon the rates of these operations. It is impossible to represent them by a single equation, and all we can hope to do is to write equations that represent the general tendencies during the two processes of charge and discharge. Starting with the cell fully charged, the positive plate is coated with lead peroxide and the negative plate is spongy lead. During the discharge the hydrogen ions flow toward the positive plate and cause a reduction of the lead peroxide:

$$PbO_2 + 2H \rightarrow PbO + H_2O.$$

A partial reaction of this lower oxide with sulphuric acid results in the formation of some lead sulphate as follows:

$$PbO + H_2SO_4 \rightarrow PbSO_4 + H_2O.$$

The SO_4 ion goes in the opposite direction toward the lead plate and, as in the case with platinum electrodes, the SO_4 reacts with water and liberates oxygen.

$$SO_4 + H_2O \rightarrow H_2SO_4 + O.$$

The oxygen thus liberated unites with the lead to form one or both of the oxides.

$$Pb + O \rightarrow PbO.$$

As at the other plate, there is also formed some lead sulphate.

$$PbO + H_2SO_4 \rightarrow PbSO_4 + H_2O.$$

In so far as these equations can represent the actual reactions, two things are to be observed: first, the tendency is to reduce the two plates to the same condition, in which state the cell is discharged and the EMF falls to a low value; second, some of the acid is removed from the solution in forming the insoluble lead sulphate which is deposited on the plates. Thus it is seen that a decrease in density of the electrolyte occurs when the cell is more or less discharged; hence it is customary to ascertain the condition of an automobile battery by a specific gravity test of its electrolyte. If the hydrometer shows a low specific gravity, it is assumed that the battery needs charging. During the process of charging a storage battery the current through it is reversed. Oxygen instead of hydrogen appears at the positive plate, the lead sulphate is oxidized, and lead peroxide again covers the surface. At the negative plate hydrogen causes a reduction of the lead oxide and lead sulphate, thus restoring approximately the initial condition.

299. The Edison Storage Battery.—The great weight of lead cells practically prohibits their use in motor vehicles propelled by electric power. The Edison battery was designed for service requiring a lighter battery than can be made from lead cells. The positive plate consists of an oxide of nickel packed into a steel grid, while the negative plate is of finely divided iron also packed into a grid. The electrolyte is a solution of potassium hydroxide. When the cell is furnishing current, the nickel oxide suffers reduction and the iron of the negative plate is oxidized. During the process of charging the cell the reactions are reversed. The electromotive force is about 1.25 volts, while that of the lead cell is two volts. Its efficiency is also lower than that of the lead cell. It may be charged rapidly or completely discharged without damage, and its rugged construction makes damage from mechanical shock very unlikely.

300. Ohm's Law.—In 1826, Dr. G. S. Ohm discovered that the current flowing through a metallic conductor is strictly proportional to the potential difference maintained between its terminals, provided there is no source of EMF in the conductor. This is equivalent to the statement that as the potential difference is made to change, the current also changes in such a way that the ratio of the potential difference to the current remains constant. Writing this in the form of an equation, we have

$$\frac{V_1 - V_2}{I} = R,$$

in which R is a constant, independent of either current or potential difference and determined only by the properties of the conductor itself. It is called the resistance of the conductor. If we multiply both members of the equation by I/R it takes the form,

$$I = \frac{V_1 - V_2}{R},$$

or, clearing of fractions,

$$V_1 - V_2 \doteq IR.$$

The relationship represented by any one of these equations is known as Ohm's law. If the conductor is changed in some way so as to alter R while $V_1 - V_2$ is kept constant, it is apparent from the equations that I varies inversely as R. This explains why R is called resistance. If water were flowing through a pipe, an increase of the resistance to the flow would cut down the current; so it is natural to use the same name for the property of an electrical conductor whose increase causes a decrease of current.

301. The Unit of Resistance.—In the practical system the unit of resistance is appropriately named the *ohm*. Since the ampere and the volt have been defined in §§278 and 284, the relationship, $I = \frac{V_1 - V_2}{R}$, furnishes the definition of the ohm. An *ohm* is the resistance of a conductor such that, when there is a potential difference of one volt between its terminals, a current of one ampere flows through it.

302. International Units.—The matter presented in this topic is of historic interest and has been and is yet of practical importance, but it is not recommended that the student burden his memory with the definitions of international units.

In §254 and again in §278 it is stated that the coulomb is equal to 3×10^9 esu. Since the esu is defined by means of Coulomb's law, it is evident that measurements in terms of this unit might involve operations that it would be extremely difficult to carry out. On the other hand, the operation of weighing the metal deposited in electrolysis is both easy and accurate. From the discussion in §283 it appears that a coulomb is conveyed through a solution of silver nitrate by a definite mass of silver. This mass has been determined several times by elaborate experiments of great precision, and is found to be very nearly .001118 gram. This number is used in formulating the definition of the international coulomb, which has been accepted by international agreement as a standard. An *international coulomb* is the quantity of electricity which, when passing through a neutral solution of silver nitrate in water, deposits .00111800 gram of silver.

An *international ampere* is the unvarying electric current which, when passing through a solution of silver nitrate in water, deposits silver at the rate of .00111800 of a gram per second.

The international congress that formulated these definitions recognized also the difficulties attending the process of measuring resistance in terms of the ohm as defined in §301, and undertook to establish a definition of that unit which would be more easily realized.

An *international ohm* is the resistance offered to an unvarying electric current by a column of mercury at the temperature of melting ice, 14.4521 grams in mass, of a constant cross-sectional area, and of length 106.300 centimeters.

Experience has shown that the mercury ohm has not been widely used. People needing a standard resistance have found it easier to procure a carefully measured wire resistance from the Bureau of Standards than to set up for themselves a mercury standard.

Further information concerning electrical units may be obtained from the *Standards Yearbook* published by the Bureau of Standards. The following quotation from the *Yearbook of 1929* shows the present attitude of officials of the bureau toward this matter and the desire to discard the international units and return to the absolute c.g.s. units as primary standards in electrical measurements.

"The American advisory committee met at the Bureau of Standards on June 16, 1928, together with a number of members of the staff of the bureau. After due consideration of the information available regarding the present status of electrical measurements, the committee unanimously adopted the following resolutions:

"*Resolved:* That in the opinion of this committee, in view of improvements which are being made in absolute measurements, electrical standards should in future be based upon the absolute system of units." (There was a second resolution not quoted here.)

In February, 1929, the International Advisory Committee made a report to the International Committee of Weights and Measures which was in entire agreement with the recommendations of the American Advisory Committee. In the summer of

1929 the recommendations of this committee will probably become authoritative through the action of the International Committee of Weights and Measures.

303. Kirchhoff's First Law.—In 1847, Kirchhoff formulated two laws which have come to be considered as classic in electrical science. Their usefulness in the solution of difficult problems justifies one in giving them considerable study. The first law is stated as follows: "If several conductors meet at a point, the algebraic sum of all the currents flowing toward the point is zero." In a steady state, since there is no accumulation of electricity at a point, the sum of the currents flowing away from a point, such as P in Fig. 175, must be equal to the sum of the currents flowing toward it. Then $I_1 + I_2 = I_3 + I_4 + I_5$. Considering I_3, I_4, and I_5 as equivalent to $-I_3$, $-I_4$, and $-I_5$ flowing toward the point P, we have $I_1 + I_2 - I_3 - I_4 - I_5 = 0$, or $\Sigma I = 0$, which is equivalent to the verbal statement above.

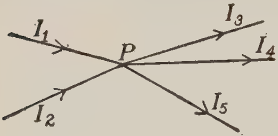


FIG. 175

304. Kirchhoff's Second Law.—In a closed circuit the total fall of potential is equal to the total rise of potential, as was stated in §285; hence it follows that if $V_1 - V_2$ represents in general the fall of potential along any one portion of the circuit,

$$\Sigma (V_1 - V_2) = \Sigma E,$$

in which ΣE represents the algebraic sum of all the EMF's in the circuit. Substituting for $V_1 - V_2$ its value taken from §300, we have

$$\Sigma IR = \Sigma E.$$

This is known as Kirchhoff's second law; stated in words it is as follows: In any closed circuit, such as *cbgh* in the network of Fig. 177, the sum of the products of the resistance of each part into the current flowing through that part is equal to the algebraic sum of the EMF'S in that circuit. The law is perfectly general; it fits any case, however complicated, and may be used in solving problems related to intricate networks. In a simple system in which all conductors are in series, as in Fig. 176, the current that flows in any one conductor is the same as that in any other; hence in the expression ΣIR, I is a factor common to all terms of the series, and $\Sigma IR = I\Sigma R$. Substituting this in the former equation and solving for I, we have

$$I = \frac{\Sigma E}{\Sigma R}.$$

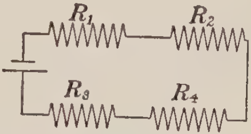


FIG. 176

This is usually considered merely as a statement of Ohm's law extended to apply to the circuit as a whole. The equation is often written

$$I = \frac{E}{R + r},$$

with the understanding that E is to be interpreted as the algebraic sum of the EMF's and R as the sum of the resistances of the conductors not including that of the electrolyte in the battery. r is the internal resistance, *i.e.*,the resistance of the battery.

When the system is more complicated and we wish to find the current in each branch, Kirchhoff's laws become indispensable. For example, consider the system shown in Fig. 177. Two cells, having EMF's E_1 and E_2 respectively, and resistances r_1 and r_2 respectively, furnish current through two wires in parallel having resistances R_3 and R_4 respectively. There will be four different currents, which may be designated as I_1, I_2, I_3, and I_4, and our problem is to determine these currents. There are six different closed circuits represented in the figure, three of which are independent. We may therefore write three independent equations applying Kirchhoff's second law to these circuits. For the circuit *achf*, the second law furnishes the equation

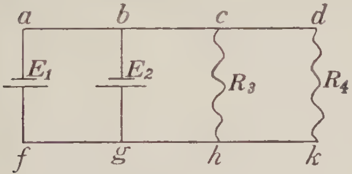


FIG. 177

$$E_1 = r_1I_1 + R_3I_3.$$

It is necessary to assume a positive direction for each current and write the equation in accordance with this assumption; then if one or more of the currents in the solution proves to have a negative sign, the current is known to have a direction opposite to that assumed. Let the reader now write Kirchhoff's second law for circuits *adkf* and *abgf*. These will be independent equations. An equation for circuit *bchg* would not be independent of the equations already written, since it could be derived from them. We have some choice as to which equations we shall use. To obtain the necessary fourth equation, we make use of Kirchhoff's first law; thus,

$$I_1 + I_2 - I_3 - I_4 = 0.$$

Having four independent simultaneous equations, we may solve for the four unknown quantities.

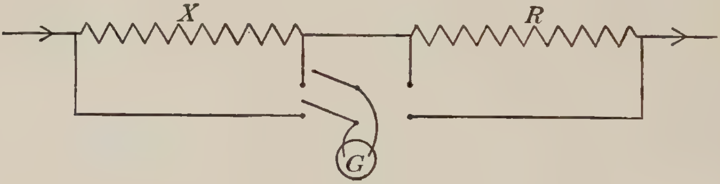


Fig. 178

305. Comparison of Resistances. Fall of Potential Method.—We shall assume for the present that it is possible to make a galvanometer whose deflection is proportional to the difference of potential between the points to which it is connected, and that this potential difference is not altered when the galvanometer is connected to the points. In Fig. 178, let G be such a galvanometer connected as shown to a double pole double throw switch. When the blades of the switch are thrown to the right, the galvanometer is connected to the terminals of a standard resistance R. If a current I is flowing through this resistance, the potential difference between its terminals is RI volts. When the switch is thrown to the left, the potential difference applied to the galvanometer is XI, X being an unknown resistance. Let D and D' be the two deflections produced; then, since they are proportional to the two potential differences,

$$RI : XI = D : D'.$$

The current is the same in the two conductors; hence I cancels from the equation, and

$$X = \frac{D'R}{D};$$

thus X is expressed in terms of known quantities. This is a standard method for the comparison of very small resistances. When the resistances are large, the current through the galvanometer cannot be neglected, and the computation of the unknown resistance becomes more complicated. Measurements by Wheatstone's bridge, described in §307, are then found to be more satisfactory.

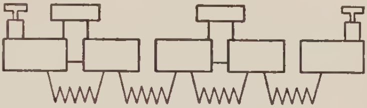


FIG. 179

306. The Resistance Box.—To construct a variable resistance, a box with a hard rubber top is used. A number of brass blocks are placed in a row a short distance apart and screwed to the rubber. Connecting each pair of adjacent blocks there is a coil having a definite known resistance, as shown in Fig. 179. The coils are under the rubber cover inside the box, while the brass blocks are on the upper side of the cover. A hole is drilled between each pair of blocks in such a way that when a brass plug is inserted in the hole it makes contact with both blocks; thus when the plug is in place an electric current can get from one block to the next without going through the coil connecting the blocks. The resistance between the terminals of the box is considered as zero when all the plugs are in place; and when some of them are out the resistance is the sum of the resistances through which the current must go in passing from one terminal to the other. Boxes such that any resistance between .1 ohm and 10,000 ohms is available are to be found in the laboratory.

307. Comparison of Resistances by Wheatstone's Bridge.—This is the method that is used more than any other; it is simple, convenient, and may be made very accurate when necessary. Fig. 180 is a diagram of the connections. A battery furnishes a current I, which at A divides into two parts, I_1 through the upper conductor, and I_2 through the lower. Any point, such as P, in the upper branch will have a potential less than that of A and greater than that of B, and there will necessarily be some point in the lower branch having the same potential as P. To find this point a galvanometer is used. One of its terminals is connected to the point P, and the other is moved along the lower conductor until the galvanometer shows no deflection. Call the point on the lower conductor thus found, Q. P and Q must have the same potential, otherwise there would be a current through the galvanometer. Let the resistances of the four parts, AP, PB, AQ, and QB, be R_1, R_2, R_3, and R_4 respectively. The difference of potential between A and P is R_1I_1 and that between A and Q is R_3I_2. Since P and Q have the same potential,

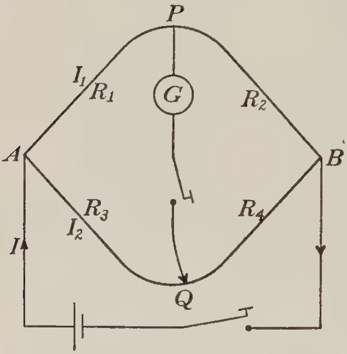


Fig. 180

$$R_1I_1 = R_3I_2.$$

By Kirchhoff's law, when no current flows through the galvanometer the current in AP equals that in PB, and the current in AQ equals that in QB. Since the fall of potential from P to B is the same as the fall from Q to B,

$$R_2I_1 = R_4I_2.$$

Dividing these equations member by member,

$$\frac{R_1I_1}{R_2I_1} = \frac{R_3I_2}{R_4I_2}, \quad \text{and} \quad \frac{R_1}{R_2} = \frac{R_3}{R_4}.$$

If three of these resistances were known, the fourth could be computed from the equation; or if one were known and also the ratio of two others, the fourth could be computed. In the slide wire bridge shown in Fig. 181 the contact Q is moved along the wire to balance the bridge, and the resistances of the two parts of the bridge wire are taken as proportional to their lengths; then

$$\frac{R_1}{X} = \frac{L_1}{L_2}.$$

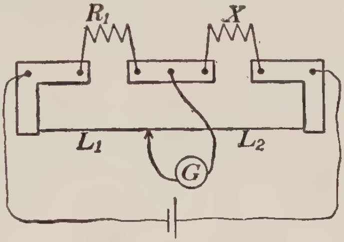


FIG. 181

In another form of Wheatstone's bridge, illustrated in Fig. 182, three plug boxes are used, and one of them is adjusted until the galvanometer shows no deflection. After this adjustment has been made, the resistances are related as before, and

$$\frac{R_1}{R_2} = \frac{R_3}{X},$$

from which

$$X = \frac{R_2}{R_1} \cdot R_3.$$

308. Laws of Resistance.—Accurate measurements of the resistance of conductors have led to the following conclusions:

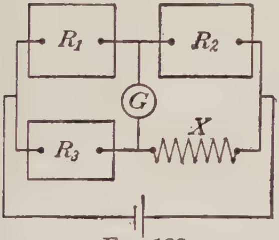


FIG. 182

(1) The resistance of a conductor of uniform cross-section is proportional to its length.

(2) For a given length the resistance of a conductor is inversely proportional to the area of its cross-section.

(3) The resistance of a conductor depends upon the material of which it is made and upon its temperature.

Calling the length of the conductor l and its cross-sectional area a, the resistance at a given temperature is expressed by the equation

$$R = \rho \frac{l}{a},$$

in which ρ is the factor that depends upon the material from which the conductor is made. The interpretation of ρ depends upon the units in which l and a are measured. If l is in cm and a in cm^2, ρ is called the *resistivity* of the substance and is numerically equal to the resistance between opposite faces of a cube of the substance one cm on each edge.

Having a table of the resistivities of different substances at hand, it is easy to compute the resistance of any ordinary conductor of given dimensions.

309. Temperature Coefficient.—The resistance of pure metals and of most alloys increases with rise of temperature; the resistance of carbon and of electrolytes decreases with rise of temperature. The latter substances are said to have negative temperature coefficients. For pure metals the increase of resistance per degree is so nearly constant for a considerable range that we are able to represent the resistance by an equation similar to that for the volume of a gas: $R = R_0 (1 + \alpha t)$. α is called the temperature coefficient, and is defined as the fraction of the resistance at 0° C. that the resistance changes for each degree change of temperature. It is a striking fact that the coefficient α is not far different from the coefficient of expansion of a gas. For copper the coefficient is about .0042 per degree, while for silver and platinum it is somewhat less. It will be remembered that the coefficient of expansion of a gas is about .00367. If the resistance temperature coefficient were exactly the same as the coefficient of expansion of a gas, and if the equation were true for all temperatures, it is evident that as the zero of the gas scale (absolute zero) is approached, the resistance would approach zero. Professor Kamerlingh Onnes has investigated the resistance of some metals at temperatures estimated to be within three degrees of the absolute zero, and has found in the case of mercury that the resistance is less than one ten-millionth part of that at the freezing point of water. Similar results were obtained with lead and tin.

Many alloys have temperature coefficients much lower than those of the pure metals. For German silver the coefficient is about 1/10 that of copper, and for manganin it is only 1/4,000 of that of copper. Because the resistance of manganin is so nearly independent of the temperature, it is used in the construction of standard resistance coils.

The following experiment illustrates the fact that the resistance of a conductor is a function of its temperature. A coil of fine copper wire is put in series with a small electric lamp, and a current is supplied to the lamp through the wire from several dry cells. The number of dry cells is chosen so that the light is very dim. If now the coil of copper wire is put into a flask of liquid air, its temperature is reduced to about −180° C., its resistance becomes less than one-third of its former resistance, and the current through the lamp is greatly increased, so much so that the lamp is brilliantly lighted. The experiment may be varied by starting with the lamp brilliantly lighted at room temperature. When the copper coil is heated over a flame the resistance is increased and the light grows dim.

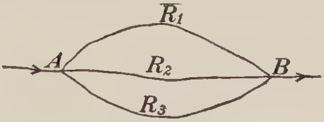


FIG. 183

310. Resistances in Series and in Parallel.—When several conductors are connected in series, the total resistance is the sum of the resistances of the conductors taken separately. However, when they are connected in parallel, as shown in Fig. 183, the combined resistance is less than that of any one of the conductors, and may be found as follows: A and B are the junction points of the several conductors. When a current I enters the group of conductors at A, it divides into parts which may be designated as I_1, I_2, I_3, etc. R_1, R_2, R_3, etc., are the respective resistances of the individual conductors. Let E represent the potential difference between the two points A and B. This potential difference is applied to each of the conductors; hence $I_1 = \frac{E}{R_1}$, $I_2 = \frac{E}{R_2}$, $I_3 = \frac{E}{R_3}$, etc. R is taken to represent the resistance of the system and is the quantity we wish to find; then $I = \frac{E}{R} = I_1 + I_2 + I_3 +$ etc. Substituting the values of I_1, I_2, I_3, etc., from the former equation, we have

$$\frac{E}{R} = \frac{E}{R_1} + \frac{E}{R_2} + \frac{E}{R_3} + \text{etc.}$$

Finally, canceling E, the value of R is given by the following equation:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \text{etc.}$$

The reciprocal of the resistance of a conductor is called its conductance. This last equation shows that the conductance of a system of parallel conductors is equal to the sum of the conductances of the individual conductors.

When there are only two conductors in parallel,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}, \quad \text{and} \quad R = \frac{R_1 R_2}{R_1 + R_2}.$$

If $R_1 = R_2$,

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{2}{R_1},$$

hence

$$R = \frac{R_1}{2}$$

i.e., two equal conductors in parallel have half the resistance of one. Similarly we see that three equal conductors in parallel have one-third the resistance of one.

311. Division of Current in Parallel Conductors.—When only two conductors are considered,

$$I_1 = \frac{E}{R_1}, \quad \text{and} \quad I_2 = \frac{E}{R_2}.$$

Dividing member by member,

$$\frac{I_1}{I_2} = \frac{R_2}{R_1},$$

which shows that the current divides into parts that are inversely as the resistances. A simple transformation of this proportion gives the following equation:

$$\frac{I_1}{I_1 + I_2} = \frac{R_2}{R_1 + R_2};$$

but $I_1 + I_2 = I$, the total current; hence, making this substitution,

$$I_1 = \frac{R_2 I}{R_1 + R_2} \quad \text{and} \quad I_2 = \frac{R_1 I}{R_1 + R_2},$$

which are the parts of the total current flowing in the respective branches. Sometimes it becomes necessary to use a galvanometer or an ammeter for measuring currents that are too large to be registered by the instrument. In such a case a wire called a *shunt* is put in parallel with the instrument and a definite fraction of the total current, determined by the above equation, passes through it. By this means it is possible to increase considerably the range of a measuring instrument. For example, if the shunt has a resistance equal to one-ninth that of the ammeter, all readings of the instrument must be multiplied by ten to obtain the total current.

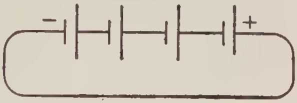


FIG. 184

312. Cells in Series.—When several cells are used to obtain a larger current than is supplied by a single cell, it should be remembered that the current may be increased either by increasing the EMF in the circuit or by decreasing the resistance. The cells are put in series by connecting the positive pole of the first cell to the negative of the second and the positive of the second to the negative of the third, and so on. The external circuit is connected between the negative terminal of the first cell and the positive terminal of the last, as shown in the diagram of Fig. 184. If n similar cells are used, the EMF of the battery is n times the EMF of a single cell, but so is the internal resistance n times that of a single cell. In accordance with Ohm's law, the expression for the current is

$$I = \frac{nE}{R + nr},$$

in which E is the EMF of one cell and r is its resistance. If R is very large, nr may be neglected, and

$$I = \frac{nE}{R},$$

which signifies that the current produced by n cells is n times as large as that from one cell. If, however, we wished to produce a large current through a short piece of wire having a very small resistance, R would be negligible as compared with nr, and the current would be approximately represented by

$$I = \frac{nE}{nr} = \frac{E}{r},$$

i.e., the current from n cells is not appreciably larger than that produced by a single cell. This indicates that the series method of connecting cells should be employed if one wishes to produce a maximum current in a conductor of large resistance.

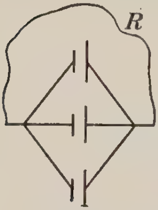


FIG. 185

313. Cells in Parallel.—A battery is sometimes made by connecting the positive terminals of all cells together and the negative terminals together. Fig. 185 is a diagram of such a battery with its external circuit. If the cells are all alike, the EMF of the battery is that of one cell, and the internal resistance is $1/n$ of the resistance of a single cell. The current in the external circuit is then

$$I = \frac{E}{R + r/n}.$$

If R were so large that r/n could be neglected, the expression for the current would be $I = E/R$, which is the same as if only one cell were used; hence there is no advantage in using several cells in parallel when the external resistance is large. On the other hand, when R is very small and may be neglected,

$$I = \frac{E}{r/n} = n\frac{E}{r},$$

which is n times the current delivered by a single cell. The conclusion to be drawn is that, to produce large currents, cells should be connected in series when the external resistance is large and in parallel if the external resistance is very small. When the cells do not have equal EMF's and equal resistances, the problem of finding the current in each cell and in the wire is solved with the help of Kirchhoff's laws.

314. The Series-Parallel Connection of Equal Cells.—When the external resistance is small but not extremely small, a combination of the series and the parallel connection may give the largest current. Suppose we have twelve cells. They might be made into three groups of four in series and the three groups put in parallel, as in Fig. 186. The equation for the current is

$$I = \frac{4E}{R + 4r/3}.$$

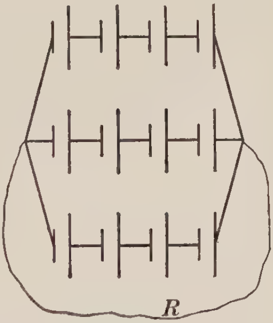


FIG. 186

In general, if there are n cells in series and m parallel groups,

$$I = \frac{nE}{R + \dfrac{nr}{m}}.$$

It may be shown that a given number of cells in equal groups will furnish the maximum current if they are so connected that the internal resistance of the battery is equal to the external resistance R. When this condition is realized the efficiency is only 50 per cent. The student should sketch two other series-parallel arrangements of these twelve cells and write an expression for the current in each case.

315. The Potentiometer.—The potentiometer is an electrical measuring device of such importance that its principle must be mastered by every student of electricity. We shall discuss here its use in a particular problem, viz., the comparison of two EMF's. A battery *B* in Fig. 187 furnishes a steady current *I* through a long straight wire of rather high resistance. The point *A* is at higher potential than any point between *A* and *D*, and hence any conductor having one end connected to the wire at *A* and the other end connected at some other point, as *P*, would carry a current flowing toward the right. If, however, a cell were put into this latter circuit, with its positive terminal connected to *A*, the EMF of this circuit would be in opposition to the current in the lower conductor. For some position of the contact *P*, such as P_1, the potential difference between *A* and *P* will be exactly equal to E_1, the EMF of the cell, and no current will flow through the cell. A galvanometer put into the circuit will show no current when this balance is secured. Next we throw the switch *S* over till the other cell is in the circuit and then adjust the position of the contact *P* till the galvanometer shows no deflection. Suppose P_2 to be this new position of *P*; then E_2, the EMF of the second cell, equals the fall of potential along the straight wire between *A* and P_2. Now this fall of potential along the wire is proportional to the resistance of the part included between *A* and *P*, since it equals *IR* and *I* is constant. Furthermore, the resistance is proportional to the length; hence it follows that

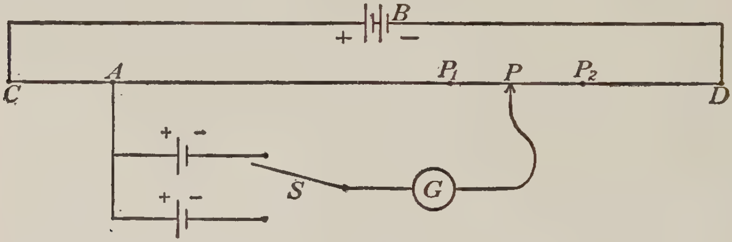


FIG. 187

$$E_1 : E_2 = AP_1 : AP_2.$$

A scale is provided for measuring the lengths AP_1 and AP_2. One of the cells may be a standard cell, *i.e.*, one whose EMF is accurately known, in which case we may solve the equation to determine the EMF of the other cell.

Review of Electrical Units

Physical Quantity	*Defining Equation*	*Practical Unit*
Quantity of electricity......	($F = qq'/r^2$ in the e.s. system)..........	Coulomb = 3×10^9 esu
Current..........	$I = Q/t$............	Ampere = $\dfrac{\text{coulomb}}{\text{sec.}} = 3 \times 10^9$ esu
Potential difference and EMF......	$V_1 - V_2 = W/Q$..... $\text{EMF} = W/Q$.......	Volt = $\dfrac{\text{joule}}{\text{coulomb}} = \dfrac{1}{300}$ esu
Field intensity....	$E = F/Q$...........	Volt per cm = $\dfrac{1}{300}$ esu
Capacitance......	$C = Q/V$...........	Farad = $\dfrac{\text{coulomb}}{\text{volt}} = 9 \times 10^{11}$ esu
Resistance.......	$I = \dfrac{E}{R}$, or $R = \dfrac{E}{I}$.....	Ohm = $\dfrac{\text{volt}}{\text{ampere}} = \dfrac{1}{9 \times 10^{11}}$ esu

PROBLEMS

1. The charge of the electron is 1.591×10^{-19} coulomb, and the mass of a silver atom 1.78×10^{-22} gram. Find the electrochemical equivalent of silver.

2. (a) The atomic mass of silver is about 108 and that of zinc 65. Using the result of problem 1, find the electrochemical equivalent of zinc. (b) How much zinc will be deposited in an electrolytic cell by 5 amperes in 10 minutes?

3. Two platinum electrodes are placed in dilute sulphuric acid, and a current of 8 amperes is sent through the solution for 10 minutes. What substances are liberated at the electrodes, and how much of each?

4. If, in the preceding problem, the electrodes had been made of copper, what would happen at each electrode? Would the plates constituting the electrodes change in mass? If so, how much?

5. How much water will 5 amperes decompose in one hour?

6. If three Daniell cells in series are used to do some silver plating, how much zinc would be dissolved in the battery for each gram of silver deposited? What would happen to the copper electrode of each cell?

7. Two electrolytic cells containing zinc sulphate solutions are alike, except that one has platinum electrodes and the other has zinc electrodes. The same potential difference is applied to both cells. Will the currents in the two cells be equal? Explain your answer fully.

8. If ten amperes flow through a solution of potassium sulphate for half an hour, the electrodes being of platinum, what four substances are produced, and how much of each?

9. A man, knowing that under certain conditions a dry cell will deliver 30 amperes, decided to charge a storage cell by using the current from a dry cell. Explain why he failed. What probably happened?

10. You have learned that 96,500 coulombs will deposit one gram-equivalent of silver. Describe an experiment by which you could prove this to be true.

11. Three wires having resistances 20, 40, and 60 ohms, respectively, are connected in series across a 110-volt lighting circuit. Find the current in each wire, and the potential difference between the terminals of each wire.

12. If the wires of problem 11 were connected in parallel, what current would flow in each wire? Find the resistance of a fourth wire which, if connected across the lighting circuit, would take the same current as the three wires together.

13. Three wires of the same material and having the same diameter are connected in parallel, and a current is then sent through them. One of the wires is 60 ft. long, the second is 90 ft. long, and the third carries one-quarter of the current. How long is the third wire?

14. The resistivity of copper is 1.7×10^{-6} ohm-centimeters. Why is it expressed in ohm-centimeters rather than in ohms per centimeter or in some other way? Find the resistance of a copper wire 100 meters long and one mm in diameter. Indicate the result without solving for the numerical value.

15. If we write $R = kL/d^2$ for the resistance of a wire L feet long with a diameter d measured in thousandths of an inch, interpret the constant k. For copper the numerical value of k is about 10.5. It depends upon the purity of the copper and varies with the temperature. Find the resistance of a wire 1,000 ft. long and 1/10 inch in diameter. d^2 measures the area of cross-section of the wire in circular mils. A circular mil = .7854 square mil. A mil is a thousandth of an inch.

16. If the resistivity of iron is taken as 12×10^{-6} ohm-centimeters, compare the diameters of two wires of the same length and the same resistance, one made of iron and the other of copper.

17. From the data of the preceding problems, compute the value of k for iron, and then find the resistance of an iron wire 100 ft. long and 10 mils in diameter.

18. If a wire is drawn out till its length is doubled, how does its resistance change?

19. How many cells in series will produce a current of 3 amperes in a wire of 1.2 ohms resistance, if each cell has an internal resistance of .3 ohms and an EMF of 1.5 volts?

20. A battery shows a potential difference between its terminals of 12 volts when it is furnishing no current, and 9 volts when its terminals are connected by a wire of 6 ohms resistance. Find the internal resistance of the battery.

21. AB is a wire of 3 ohms resistance; BC is a wire of 4 ohms resistance. A third wire of 5 ohms resistance connects A and C, thus forming a divided circuit. A current of 5 amperes flows in at A and out at C. Find the current in each of the three wires. Make a sketch.

22. In the preceding problem, find the potential difference between A and B and between B and C. A fourth wire is connected to B; at what point could its other end touch the third wire without causing a current in the fourth wire?

23. Five coils of wire have resistances 3, 4, 5, 6, and 8 ohms of resistance, respectively. They are connected to form three parallel conductors. One branch consists of the 3 and 4 ohm coils in series; another of the 6 and 8 ohm coils in series, and the third branch is the 5 ohm coil. Sketch the system and compute its resistance.

24. If 8 amperes are sent through the divided circuit of problem 23, what current will flow in each branch?

25. Calculate the resistance of a shunt for a galvanometer of 900 ohms resistance, such that .05 of the current will go through the galvanometer.

26. Eight cells, each having an EMF of 1.6 volts and internal resistance of 1.8 ohms, are used to furnish current through a wire of 3.6 ohms resistance. How should they be connected for maximum current? What is the value of the maximum current?

27. Four dry cells, A, B, C, D, each having an EMF of 1.5 volts, are connected in series to a wire of 10 ohms resistance. A and B each have an internal resistance of .1 ohm, C has a resistance of .2 ohm, and D has a resistance of 12 ohms. What current will be produced? If *D* were omitted from the circuit, what current would be produced?

28. Make diagrams of three ways three cells could be connected to a wire of resistance *R*, and write a formula for the current in each case.

29. A cell has an internal resistance of 2 ohms. A voltmeter of 100 ohms resistance is used to measure its EMF, the reading being 1.4 volts. What is its true EMF? What kind of a voltmeter would give a more accurate result?

30. A storage battery whose EMF is 24 volts produces a current of 8 amperes through a wire of 2.4 ohms resistance. What current would it produce through a wire of 4.2 ohms resistance?

31. Sketch the following system: A cell having an EMF of 1.6 volts and resistance of .2 ohm furnishes current through a divided circuit, the two branches of which have resistance of 3 and 5 ohms respectively. Write Kirchhoff's second law for each of the three closed circuits. Show that one of these equations can be derived from the other two, and that therefore there are only two independent equations given by the second law.

32. In the preceding problem, write Kirchhoff's first law for the system, and compute the three currents.

33. A cell having an EMF of 1.6 volts and resistance of .2 ohm is connected in parallel with a cell having an EMF of 2 volts and resistance of .1 ohm. The external circuit is a wire having a resistance of .4 ohm. Using Kirchhoff's laws, find the current in each cell and the current in the wire. Can you tell what would happen if the external resistance were made smaller? If it were made larger?

34. Each cell of a battery has a resistance of .4 ohm and an EMF of 1.5 volts. For what external resistance would the battery furnish the same current for the parallel as for the series connection? What is the effect of changing the EMF?

35. Two wires are connected in series across a 220-volt power circuit, and the current is found to be 3 amperes. The same wires, when connected in parallel across the same power circuit, take a current of 15 amperes. What are the resistances of the wires?

36. To increase the range of an ammeter having a resistance of .0075 ohm, it is shunted with a resistance of .0025 ohm. When the instrument reads 12 amperes, what is the total current in the circuit? Sketch the system.

37. Enumerate all the resistances that can be obtained from three coils of resistances 3, 5, and 7 ohms, respectively, by the various ways in which they may be connected, all three coils being always in use.

CHAPTER XXI

WORK, ENERGY, AND POWER

316. Joule's Law.—The fact that a conductor is heated when it carries an electric current is familiar to everybody. Many electrical appliances, including the electric light, make use of heat thus developed. The relation of the amount of heat to the current producing it and to the resistance of the conductor was investigated by Joule in 1840. He showed that the amount of heat developed in a conductor is given by the following equation, in which H is the number of calories developed:

$$H = .24\,I^2\,R\,t.$$

This relationship is known as Joule's law.

The .24 is the reciprocal of the mechanical equivalent of heat. It will be of interest to see that we can derive this equation from definitions and laws with which we are already familiar. In the practical system of electrical units the energy unit is the joule, which is 10^7 ergs. By Ohm's law,

$V_1 - V_2 = IR$ joules per coulomb used in causing the current to flow and converted into heat in the conductor.

$(V_1 - V_2)Q = IRQ$ joules for Q coulombs.

$Q = It$, when I is in amperes and t in seconds.

Substituting this value of Q in the second member of the last equation,

$(V_1 - V_2)Q = I^2Rt$ joules converted into heat in t seconds.

One calorie = 4.186 joules.

One joule $= \dfrac{1}{4.186}$ calorie = .24 calorie, approximately.

I^2Rt joules $= .24\,I^2Rt$ calories.

317. Energy and Power.—The equation for electrical energy developed in the preceding paragraph may be put in another form. By substituting E for IR in the equation,

$$\text{Energy} = I^2Rt,$$

we have

$$\text{Energy} = EIt.$$

This is expressed in joules when E is in volts, I in amperes, and t in seconds. This latter form of the energy equation is useful when the EMF rather than the resistance is known.

To obtain the expression for electric power, we recall that power is the time rate of doing work, *i.e.*, it equals the energy expended per second.

$$\text{Power} = \frac{\text{Energy}}{\text{Time}} = \frac{EIt}{t} = EI.$$

If E is expressed in volts and I in amperes, the power EI is in joules per second, *i.e.*, in watts. A 40-watt lamp on a 110-volt lighting system is using a current equal to .364 ampere.

318. The Kilowatt-Hour.—Since energy equals power times time, it is legitimate to couple a power unit with a time unit to make an energy unit; thus one joule equals one watt-second. The commercial companies that sell electrical energy for power and lighting purposes use the kilowatt-hour as an energy unit and render their bills in terms of this unit. Since there are 1,000 watts in a kilowatt and 3,600 seconds in an hour, it is evident that one kilowatt-hour is equal to 3,600,000 joules.

319. The Electric Light.—If our ears were so restricted in range that we could hear sounds lying in one octave only, most of the sounds that we now hear would be inaudible. That is precisely the kind of limitation that is placed upon our eyes. Visible light lies in a narrow band of frequencies covering about one octave. A steam radiator in a dark room is invisible not because it does not radiate energy but because our eyes do not respond to radiations of such low frequency as those which the radiator emits. As a source of illumination its efficiency is zero. If the radiator should be heated by some means to 550° C. it would begin to emit some light of a dull red color, but only a small fraction of its radiated energy would produce any effect upon the eye. As the temperature is raised still further, a larger and larger portion of the radiated energy falls in the visible region; but before the luminous efficiency of the radiator could be brought up to one per cent, the iron would melt. These rather superficial observations lead to the conclusion that so long as our source of light is a hot solid, as in the incandescent lamp, a reasonably high efficiency can be attained only by operating the filament at a very high temperature.

The filament of the first incandescent lamp was of platinum, which melts at 1755° C. Electric lighting with incandescent lamps became a commercial possibility with the introduction of the carbon filament in 1879. The first carbon lamps were very inefficient; but the processes of manufacture were soon improved and lamps with an efficiency of more than 1/4 candle power per watt were put on the market. The temperature at which these lamps operated was slightly less than 2000° C. Their efficiency could be greatly increased by raising the filament temperatures, *i.e.*, by increasing the voltage applied to them. It was found that a 10-per-cent increase of voltage would increase the illumination by 100 per cent; however, at this high efficiency, the carbon filament rapidly disintegrated, the bulb was blackened and the lamp soon burned out. For about twenty years the lighting industry was dependent upon the carbon lamp which gave an actual luminous efficiency of about one per cent, *i.e.*, only about one per cent of its radiation was in the visible region.

The melting points of most metals are below a white heat, so that only a few of the metals could be considered as possible material for lamp filaments. The fusion points of some of these are as follows: platinum 1755° C., titanium 1900°, iridium 2300°, osmium 2700°, tantalum 2850°, and tungsten 3000°.

We lack time to consider the efforts that have been made to use these different metals, and the long and brilliant researches that have finally made tungsten available in the form of fine wire for lamp filaments. The present vacuum lamp has a tungsten filament and shows an efficiency more than three times that of the carbon lamp. The limit of efficiency, as in the case of the carbon lamp, is set by the evaporation of the tungsten and the blackening of the bulb. It is found that by introducing an inert gas, as nitrogen or argon, into the bulb, this evaporation is greatly reduced and the temperature may be still further increased, thus raising the efficiency to two candle power per watt. We are getting at least six times as much light for a given expenditure of energy as was formerly possible with incandescent lights. Researches carried on in two or three physical laboratories are largely responsible for this change, the importance of which to the world it is difficult to estimate.

PROBLEMS

1. A 40-watt tungsten lamp is operated on a 110-volt circuit. How much current does it use? What is its resistance while hot?

2. If 5.5 amperes from a 110-volt lighting circuit are required to operate an electric flatiron, how much heat does it develop per minute? What is the resistance of the heating coil while hot?

3. An electric heater in a street car uses 10 amperes from the 550-volt line. Express in horsepower the power to operate it. How many BTU per minute does it develop?

4. What current would be required to operate a 20 h.p. 220-volt motor if its efficiency is 88 per cent?

5. In an electrolytic cell for the deposition of copper, a potential difference of .4 volt is maintained between the plates. If the price of electrical energy is 3¢ per kilowatt-hour, what is the cost per pound of copper deposited?

6. If the cost of electrical energy is 4¢ per kilowatt-hour, find the cost of heating electrically 10 liters of water from 10° to 100° C.

7. What current flowing in a wire of 5 ohms resistance will produce 38,000 calories per minute? What voltage must be applied to the wire?

8. An arc lamp for street lighting takes about 6.6 amperes and

has a difference of potential between its terminals of about 72 volts. Find the horse power to operate 50 arc lamps in series. What EMF would be required?

9. Solve problem 8, making the assumption that the total resistance of the line wires is 8 ohms. What is the efficiency of the transmission?

10. How much electrical energy is produced by a dry cell for each gram of zinc dissolved, if the EMF of the cell is 1.5 volts?

11. If it requires 1,000 h.p. to drive a generator which furnishes 300 amperes at 2,200 volts, find the efficiency of the generator.

12. An electric hoist operated at 550 volts and using 20 amperes can lift 2 tons 300 feet in four minutes. Find its efficiency.

13. Compute the work in kilowatt-hours required to operate a 10 h.p. motor for 30 minutes, if its efficiency is 90 per cent.

14. An electric motor is used in operating the lift in a coal mine. If it takes 25 amperes at 220 volts when lifting a load of 2 tons vertically, at what speed can it lift the load if the over-all efficiency of the machine is 60 per cent?

CHAPTER XXII

THERMOELECTRIC PHENOMENA

320. Thermoelectromotive Force.—When two wires of different materials are joined together at their ends to form a circuit, it is found that a current usually flows in the wires if the two junctions are at different temperatures. This discovery was made by Seebeck in 1822. If iron and copper are the two metals used, the current flows from copper to iron at the hot junction and from iron to copper at the cold junction. If the temperature of the cold junction is gradually raised, the current decreases and finally becomes zero when the two temperatures are equal. This suggests that probably there is an EMF at each junction which depends upon the temperature, and when the temperatures are alike the EMF's are equal and opposite. A pair of metals thus used is called a thermocouple, and the resultant EMF in the circuit is called the thermoelectromotive force. In the case of iron and copper, as in Fig. 188, if one junction is kept at a constant temperature while the other is heated, the EMF increases rapidly at first, then more slowly, and becomes a maximum at 275° C. As the temperature of the hot junction is still further increased, the EMF begins to decrease, and finally reverses when the temperature of the hot junction is as far above 275° as that of the cold one is below it. Fig. 189 shows this relation graphically. The curve is approximately of parabolic form. The temperature 275° is called the neutral

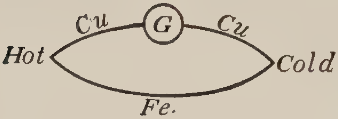


FIG. 188

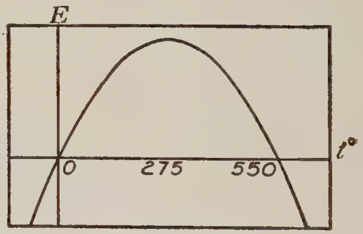


FIG. 189

point for this couple. Not all thermocouples show this reversal of the direction of the EMF. Some show very nearly a direct proportionality between temperature and EMF until the junction reaches a red heat.

321. The Peltier Effect.—An effect which is the reverse of that discovered by Seebeck was announced by Peltier in 1834. If a current from a battery is made to pass through a junction of two metals, the junction is either cooled or heated according to the direction of the current. The heat is proportional to the current and not to its square, as when heat is produced in accordance with Joule's law. Furthermore, the effect is reversible, *i.e.*, the same amount of heat is absorbed when the current flows in one direction as is evolved when it flows in the opposite direction. When heat is developed in a wire by virtue of its resistance, the process is not reversible and the amount of heat, $.24 I^2Rt$ calories, is the same whether the current flows in one direction or the other.

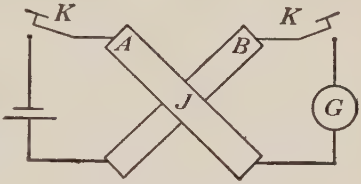


FIG. 190

The fact that a junction may be cooled by the passage of a current through it is a proof that an EMF exists at this junction and that it has a direction such as to assist in producing the current. Only by this means could the junction give up energy and be cooled by the passage of current. When the current is in opposition to this EMF, the current does work and the junction is heated. These thermal effects are superposed upon the ordinary Joule heating effect and may be masked by it so that they are not detected. This is likely to be the case if the resistance of the wires near their junction is large.

An apparatus for demonstrating the Peltier effect is shown in Fig. 190. The bars marked *A* and *B* may be of antimony and bismuth respectively. Connect the positive terminal of the battery to the bismuth, and the negative through a key K_1 to the antimony. Close K_1 for a few seconds, then open K_1

and close K_2. The galvanometer shows a deflection which soon falls to zero because the junction quickly regains its former temperature. In order to satisfy ourselves that the junction was cooled rather than heated, we may apply a little heat to it, and we then find that the deflection is opposite to that observed after the current had passed through the junction. If a number of junctions are connected in series, the effect is more easily detected.

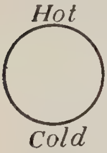


FIG. 191

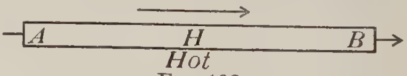


FIG. 192

322. The Thomson Effect.—Sir William Thomson (Lord Kelvin) asserted that there should be an EMF between a hot part and a cold part of a circuit consisting of only one metal. For example, there might be an EMF directed from the hot side to the cold side of the circular loop shown in Fig. 191. Such an EMF would occur in both halves of the loop, and in the circuit these would oppose one another so that no current would flow. It will be of interest to see how Thomson proved the existence of these EMF's. In Fig. 192, let AB be a portion of a circuit with a current flowing toward B. We assume that the middle portion of the conductor is heated and the ends are kept cold. If there is an EMF tending to cause current to flow from the hot portion toward the cold portions of the conductor, then the current from the battery, indicated by the arrow, is flowing against this EMF in the section AH and is therefore contributing some of its energy to this portion of the conductor. There will then be more heat in this part of the conductor than there would be if the current were not flowing. In the section HB, however, the EMF is in the same direction as the current; this part of the conductor is therefore contributing energy to the current and consequently is losing heat. The result is that there is an unequal distribution of heat on the two sides of the place where the heat is applied. This phenomenon was actually found by Thomson, and was interpreted by him as conclusive evidence of the existence of EMF's in the conductor. These EMF's are called Thomson EMF's. In some substances the action is such as to make it appear that heat is conducted more readily against the current than with it, while with others the reverse is true. It now becomes apparent that the thermoelectromotive force in a circuit consisting of two metals is really the resultant of two Peltier EMF's and two Thomson EMF's.

323. Thermoelectric Power.—Thermoelectric power is the name given to the variation of thermoelectromotive force per degree change of temperature of one of the junctions. For an iron-copper junction, the thermoelectric power for any temperature is the slope of the curve of Fig. 189 at the point corresponding to that temperature. This evidently becomes zero at the neutral point, and has a negative value for temperatures higher than that of the neutral point. Its average value for the range from zero to 275° is about .000010 volt per degree. For iron and constantan in the neighborhood of 20° C., the thermoelectric power is about .000036 volt per degree; and for an antimony-bismuth couple it is more than .000100 volt per degree. When thermocouples are used to measure small changes of temperature, the thermoelectric power may be considered as constant and the EMF is the product of the thermoelectric power and the temperature difference between the two junctions; thus, if P is the thermoelectric power,

$$E = P(t - t_o).$$

When a large range of temperatures is to be covered, P is variable and the computation of the temperature from the measured EMF is slightly more complicated. In this case it is customary to calibrate the couple over the desired range by comparing its indications with those of a standard thermometer. It is convenient to make t_o the temperature of melting ice; then if the range does not extend beyond the neutral point, one value of t and only one corresponds to each value of E. Since E can be measured with accuracy, t is determined. Commercial pyrometers for measuring the temperatures of furnaces are made with a thermocouple consisting of platinum and a rhodium-platinum alloy enclosed in a porcelain tube. The cold junction is kept in a thermos bottle containing melting ice while the porcelain tube is inserted in the furnace. The calibration is usually so made that the instrument for measuring the EMF reads the temperature directly.

324. The Thermal Cross.—The measurement of high frequency oscillating electric currents, such as are used in radio broadcasting, presents peculiar difficulties. The employment of a thermocouple together with an ordinary galvanometer for this purpose has become quite general. A thermal cross consists of two fine wires of different material crossed as shown in Fig. 193, and soldered or welded together. Suppose that iron and constantan are used. The high frequency current follows the path *abJcd* and thus passes through half the iron wire and half the constantan wire in the cross. Both these substances have high resistivity and are readily heated by the current. As the junction *J* is heated, a thermoelectric current is set up which causes a deflection of the galvanometer. The high frequency current, if sent through the galvanometer, would produce no effect, but its effect in heating the junction is the same as that of an equivalent direct current. The apparatus is calibrated by using a direct current which may be measured with any sensitive current meter. The thermal cross is most satisfactory when it is sealed in a glass tube from which the air has been removed. When the loss of heat by conduction is thus prevented the junction becomes very sensitive.

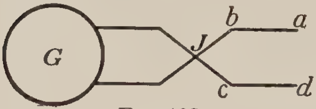


FIG. 193

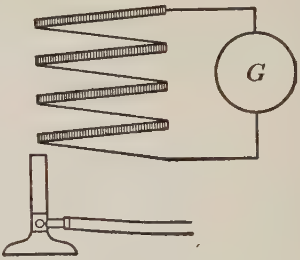


FIG. 194

325. The Thermopile.—The thermoelectromotive force of a single couple is very small; but when larger EMF's are necessary, several couples may be connected in series, just as a battery may be made up of several cells. Fig. 194 shows four thermocouples in series. The strips of different material occupy alternate positions in the circuit so that the EMF's of the four couples are added. By using a sufficiently large number of couples in series, it is possible to obtain enough current to ring an electric bell. However, it should be understood that the thermopile is seldom used for the production of an electric current where a dry cell would serve the purpose as well. It is in connection with various measuring devices that thermoelectric apparatus is chiefly useful. The radio micrometer is a type of thermogalvanometer for which the claim is made that it will detect the radiation from a lighted candle at a distance of two miles. It is with this instrument that the radiations from the stars have been compared.

CHAPTER XXIII

MAGNETISM

326. Magnets.—The phenomenon of magnetic attraction and repulsion was known to the ancients in various parts of the world even before they were aware of the similar action between electric charges. It appeared to be a magnetic property characteristic of certain stones found in Magnesia. These natural magnets consist of an oxide called magnetite which occurs frequently in many parts of the world, though not always in a magnetic condition. It usually possesses the property of attracting to itself small pieces of iron. It will be recalled that the electrified glass rod could attract small pieces of any material, not indeed with the same force but always with a force very easy to observe. The magnet attracts iron to a degree far greater than it does any other substance.

Europeans had known the natural magnet for many centuries before they discovered that when one of these bodies is free to rotate, it always places itself in a definite north and south position. Magnetic needles then became of great importance in early navigation.

It was soon found that this magnetic property could be given to pieces of hard iron not previously magnetized by stroking them with a natural magnet. Such "artificial" magnets will then also orient themselves in a north and south direction, always with the same end pointing toward the north. It is obvious, therefore, that the two ends have definitely different properties. The end which seeks the north has been called the north pole, and the other the south pole. This reminds one of the two kinds of electricity which have just been studied. There are, in fact, many similarities, but there are also striking differences. For example, it is possible to charge a body positively or negatively as we please; but a magnet may never possess one kind of pole without the other. If a thin magnet is broken in the middle, new poles will appear at the fracture so that both pieces are provided with both kinds of pole. Moreover, both poles of a magnet will always have the same strength.

327. Induced Poles.—An unmagnetized bit of iron is attracted equally by the north or south pole of a magnet. This again is analogous to the attraction of bits of paper by both kinds of electricity, and the reason is similar. The magnetic poles are surrounded by a region which is the residence of a magnetic field. If a bit of magnetic material is brought into a magnetic field, poles are induced in it. Thus, if a piece of iron is brought near to the north pole of a magnet, a south pole is induced on the near side and a north pole on the far side. Unlike poles attract and like poles repel. The piece of iron is therefore urged toward the magnet. If the south pole had been approached, the induced poles would have had opposite signs and the iron would have been attracted as before. If the iron actually touches the magnet, there is no neutralization of polarity with subsequent repulsion. This phenomenon in electricity was due to a conduction of electricity from one body to the other. There are no conductors of magnetism. Poles may not be shifted on a given body nor transferred from one body to another by *conduction.*

328. Ferromagnetism; Paramagnetism and Diamagnetism.—The only materials whose magnetic properties approach in magnitude those of iron and its compounds are the two metals, cobalt and nickel, and a very few alloys. This group is called ferromagnetic. Many other materials are very feebly magnetic. A remarkable class of these weakly magnetic materials is not attracted by a magnet but repelled by it. Chief among these materials is bismuth; another similar metal is antimony, and many gases have the same property, particularly when heated. Materials which show this reversed magnetic effect are called diamagnetic. This is a phenomenon which has no parallel in electricity; the bound charges in the atoms could never be polarized in the opposite sense and still obey Coulomb's law. The reason for this in magnetic substances will be shown later.

All materials which are attracted by magnets are called paramagnetic; those which are repelled, diamagnetic.

329. Coulomb's Law for Magnets.—We have learned that electric charges attract or repel each other with a force proportional to the product of the charges and inversely as the square of the distance between them. Gauss established a precisely similar law for the action between magnetic poles. To prove this by direct experiment was not easy in the case of electric charges. It was necessary to use very small bodies, spheres, for example, whose radii were small in comparison with the distance between the centers. Otherwise the redistribution of charge would alter the effective distance. The difficulty is even greater in the case of magnetism. Poles in magnets are at best only approximately concentrated at points, and a north pole can never be isolated. Conditions are best realized with a long thin magnet. The law has, however, been tested indirectly, and it is established with great precision. It is customarily called Coulomb's law, as it was in electricity.

330. Unit Pole.—As in the case of electric charges, we must turn here also to Coulomb's law for the definition of our unit. Unit magnetic pole is defined as that pole which will repel an equal and like pole at a distance of 1 cm with a force of 1 dyne. Again the force must be measured in the absence of all bodies or media in which poles may be induced. Strictly, the measurement should be made *in vacuo*. We may write then,

$$F = \frac{mm'}{r^2}.$$

331. Magnetic Field.—The intensity of magnetic field at a point is equal to the force in dynes which unit pole will experience if placed at that point. Magnetic field intensity is denoted by the symbol H. Its unit is the *gauss*. The magnetic field at a point has an intensity of one gauss if unit pole brought to that point will experience a force of one dyne. In general, if a pole of strength m is brought to a point at which the field intensity is H gauss, the force on it will be given by the equation,

$$F = mH.$$

The direction of the field is given by the direction of the force on a north pole.

As in the case of electric fields, it is customary to represent both the direction and magnitude of H with the help of lines of force. These lines will emerge from north poles and end on south poles. Fig. 195 shows a diagram of the lines of force due to a bar magnet. The field at any point is the resultant of the fields due to the individual poles, and is similar to the electric field due to two equal and opposite point charges.

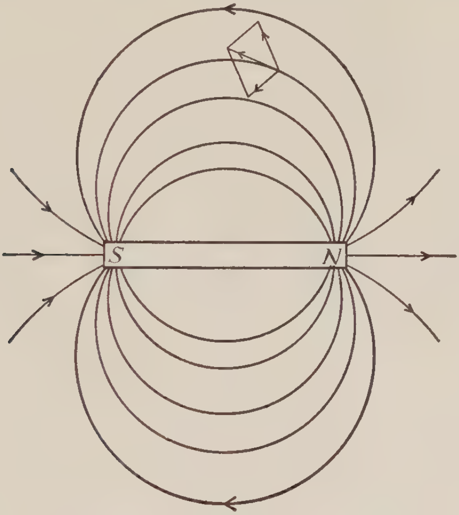


FIG. 195

A magnetic field is uniform or homogeneous when it has the same direction and magnitude at every point. The lines of force in such a field are parallel and equidistant.

332. Magnetic Moment.—If a magnet is pivoted at its center and placed in a uniform magnetic field, its two poles will be acted on by equal and opposite forces, each of which will equal mH. Such a pair of forces forms a couple. It does not tend to produce translation of the magnet as a whole, but tends to rotate it about its center of mass. If the axis of the magnet makes an angle θ with the direction of the field, the moment of the couple is

$$T = HmL \sin \theta,$$

where L is the distance between the two poles. The torque is a maximum when $\theta = 90°$, and

$$T = HmL.$$

The product mL is called the moment of the magnet, and is denoted by the symbol M. It is a property of the magnet and does not depend on strength of field.

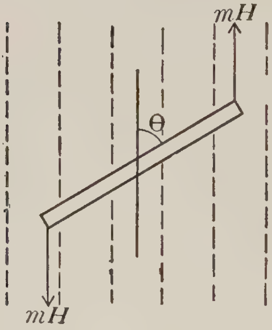


Fig. 196

In practice, the pole strength, the axis and the position of the poles in a magnet may all be unknown. There may, in fact, be no definite points at which we could locate two equivalent poles so that they would produce the same field as the actual magnet. Nevertheless, it is always possible to determine the moment of the magnet by suspending it in a known uniform field and measuring the torque necessary to hold it when it is in a position such that this torque is a maximum. Its axis is then at right angles to the field, and

$$T = HM, \quad \text{or} \quad M = \frac{T}{H}.$$

333. Terrestrial Magnetism.—A magnet, when free to turn, will set its axis in a definite orientation which coincides with the direction of the field in which it finds itself. The earth itself possesses a magnetic field and tends to orient the axes of all magnets free to move. This is the well-known principle of the compass. The earth may therefore be thought of as a huge magnet with fairly definite resultant poles. The position of these poles is only approximately constant, and the poles are not coincident with the earth's rotational poles. The axis of a compass does not therefore coincide with the geographical meridian in general. The angle between them is called the declination of the compass and is at present for Ann Arbor about two degrees to the west of north. At Washington the declination is nearly 6° to the west.

Compass needles are usually mounted on vertical axes and are rotated by the horizontal component of the earth's field. If the axis is horizontal, the position of the needle will indicate that the earth's field in this latitude has an even greater vertical component.

Fig. 197 shows roughly the relative magnitudes of these two components, and the direction of the resultant field in this latitude. The angle Θ is called the angle of dip or the angle of inclination. At Washington it is about 70°. The total intensity of the earth's field at Washington is about 0.58 gauss. In Ann Arbor the angle of inclination is about 73°. The horizontal component $H_x = 0.18$ gauss, the vertical $H_v = 0.59$ gauss.

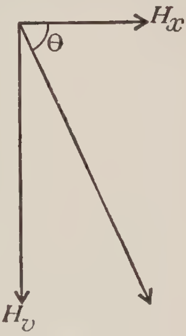


Fig. 197

PROBLEMS

1. The distance between the poles of a bar magnet is 20 cms and the pole strength is 150 c.g.s. units. Find the intensity of field at a point on the axis of the magnet 10 cms from the north pole and 30 cms from the south pole.

2. Find the intensity of field at a point 10 cms from the middle of the magnet of the preceding problem, and equally distant from its two poles. Draw the diagram.

3. If the magnet of problem 1 were placed in a horizontal position, and made an angle of 30° with the earth's meridian, find the torque tending to turn it in the horizontal plane. ($H = .19$ gauss.)

4. A bar magnet 30 cms long and of pole strength 100 c.g.s. units is placed to the west of a short needle suspended to turn in a horizontal plane. The needle lies on the axis of the magnet 10 cms from the nearer pole. How many degrees will the needle be deflected from the meridian, if $H = .19$ gauss?

CHAPTER XXIV

ELECTROMAGNETISM

334. Magnetic Field Due to a Current.—It has been pointed out in §282 that a conductor which carries an electric current is surrounded by a magnetic field. If a compass needle is held in the vicinity of a long straight wire carrying current, the needle will turn, not so that its axis is in a line which intersects the wire, but at right angles both to that line and to the wire. In other words, the lines of force due to the current do not start nor end on the conductor, but form circles about it as an axis. Such lines are shown in Fig. 198. If in this figure the current is in the plane of the page and flowing to the right, the lines of force will encircle it, going down in front and up behind. If a compass needle is held above the wire, its north pole will point toward us; if held below, the south pole will point toward us. One may find the direction of the lines of force by the following rule: If a right hand screw is turned in the direction of the lines of force, it will progress in the direction of the current. A second method is to grasp the wire with the right hand so that the thumb points in the direction of the current. The lines of force may then be drawn in the direction in which the fingers point.

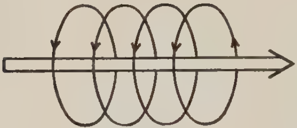


FIG. 198

The details of this field are strikingly seen by passing a wire carrying current through a sheet of paper at right angles to it and sprinkling iron filings on the paper. The small bits of iron become induced magnets and align themselves with the field. They form, therefore, almost continuous stream lines of *H* and show clearly that these lines are circular about the wire. Fig. 199 shows these lines in a plane perpendicular to two parallel conductors carrying current in opposite directions.

335. Field Due to a Circular Coil.—If the conductor is bent in the form of a loop, it may easily be seen, by applying the rule given above, that the interior of the loop is threaded by lines of force all in the same direction. In fact, this same rule may be used to determine the side of the coil from which the lines emerge. If a right hand screw is turned in the direction of the current, it will progress in the direction in which the lines emerge from the loop.

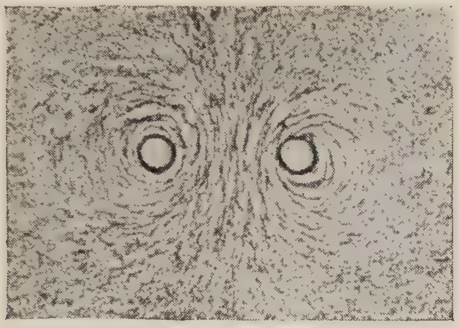


Fig. 199

In the ordinary magnet, lines of force are drawn starting from the north pole and ending on the south. The loop of current may accordingly be thought of as equivalent to a magnet whose north pole is on that side from which the lines emerge. If the loop is suspended, it may easily be shown to have all the properties of orientation and of repulsion and attraction in the presence of magnets or other current loops. Thus, if two loops are suspended parallel to each other and if current is sent through them in the same direction, they will attract each other, since the north pole of one faces the south pole of the other. If the current in one of them is reversed, they will repel each other.

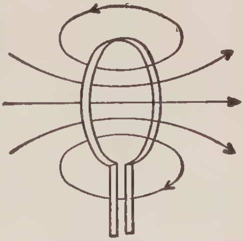


Fig. 200

336. Magnitude of the Field Due to a Current.—The relation between the magnitude of the field and the strength of the current to which it is due was first formulated by Biot and Savart. It may be expressed as a vector sum of the contributions of all the small elements of the current. Thus, the field at P due to the small current element ds, in Fig. 201, is

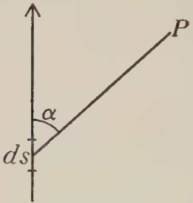


FIG. 201

$$k\frac{Ids\sin\alpha}{r^2},$$

where I is the current, r the length of the line which connects the current element ds to P, and α the angle which that line makes with ds. k is a proportionality factor which depends on the units in which the current is measured.

As an example of the use of this formula, we may calculate H at the center of a circular loop. The field at O in Fig. 202, due to any small element ds, is a vector whose direction is perpendicular to the plane of the circle. Since these small vectors are all in the same direction, their magnitudes may be added algebraically.

$$H = k\Sigma\frac{Ids\sin\alpha}{r^2}.$$

α is here the angle between the radius and tangent of the circle and is everywhere 90°; $\sin\alpha = 1$. r is the radius of the circle and is also constant. Accordingly,

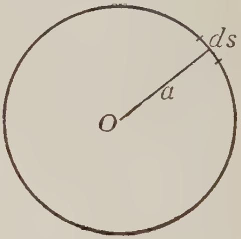


FIG. 202

$$H = k\frac{I}{r^2}\Sigma ds = k\frac{I}{r^2}2\pi r = k\frac{2\pi I}{r}. \qquad (1)$$

If the conductor had been a long straight wire, both r and α would be different for the different elements of the current, and the calculation would require the calculus. The result for a long straight wire is

$$H = k\frac{2I}{r},$$

where r is the perpendicular distance of the point in question from the wire.

337. Electromagnetic Units.—If in either of these formulas H is measured in gauss and r in centimeters, the value of k will be determined by the units in which I is measured. If $k = 1$, I is given in "electromagnetic c.g.s." units (emu). Equation (1) therefore defines the electromagnetic unit of current. Thus, if one electromagnetic c.g.s. unit (emu) of current flows through a circular circuit of 1 cm radius, the magnetic field at the center will be 2π gauss. An emu of quantity of electricity is that quantity which is transported per second by 1 emu of current. This is the third unit of quantity which has been defined. The three are related by the following equations:

1 emu of quantity = 10 coulombs $= 3 \times 10^{10}$ esu of quantity.

Also,

1 emu of current = 10 amperes.

If H is measured in gauss, r in centimeters and I in amperes,

$$k = \frac{1}{10}.$$

338. Tangent Galvanometer.—Most types of galvanometers make use of the phenomenon just described as a means of measuring current. The simplest of these in principle which allows one to make a direct measurement of current is the tangent galvanometer. It consists of a large circular coil of wire mounted in a vertical plane with a short magnet free to turn horizontally at its center. The coil is placed in the earth's magnetic meridian so that the needle stands in the plane of the coil (north and south) when no current is flowing through the coil. The field due to the current will then be east and west. If a current is sent through the coil, each pole of the magnet will be acted upon by two forces at right angles, one due to the horizontal component of the earth's

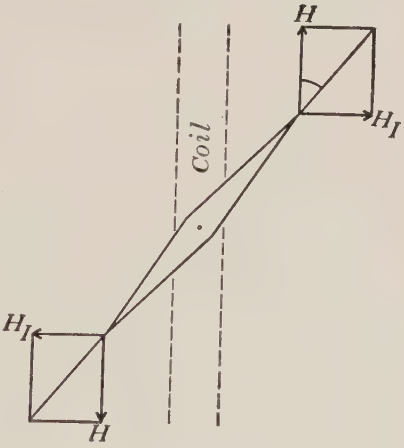


FIG. 203

field which we shall call H and the other due to the field of the current which we shall call H_{I}. The magnet will therefore take a position such that its axis lies in the direction of the resultant of these two forces. If θ is the angle which the deflected needle makes with the plane of the coil,

$$H_{\mathrm{I}} = H \tan \theta.$$

Also,

$$H_{\mathrm{I}} = \frac{2\pi I}{10r} N,$$

where I is the current in amperes, N the number of turns of wire in the coil, and r the radius of the coil. Therefore,

$$I = \frac{10r}{2\pi N} H \tan \theta.$$

This expression for H_{I} is true only for the center of the circle. The magnet must therefore be short in comparison with the radius of the coil, if the final formula is to be valid. With a different arrangement of the coil, it is possible to make the field uniform through a somewhat larger region.

339. Astatic Galvanometer.—A serious disadvantage of the tangent galvanometer is that it requires a knowledge of the horizontal component of the earth's field and, in addition to this, of the field due to any other sources in the building or outside it. Thus, masses of iron will invariably have induced poles; also, electric circuits in the neighborhood, including those of electric railways, will contribute to the resultant field which causes the "restoring couple" for the moving magnet. This resultant field is far from constant, and the instrument will usually require calibration if it is to be used for more than a comparison of currents.

A more serious difficulty lies in the fact that the circular coil carrying current must be large to give a uniform field for the needle, and the deflecting couple due to the current is therefore weak in comparison with the restoring couple. As a result, the instrument is only moderately sensitive.

To improve this, two magnets may be mounted on a single suspension. The polarities of the two magnets are opposite in direction, so that the resultant torque due to the earth's field is nearly zero. Coils are wound as shown in Fig. 204, so that both tend to turn the magnets in the same direction. The restoring torque due to the twist in the suspension is practically negligible. The deflection for small currents is consequently large and the galvanometer may be made extremely sensitive. Such a mounting is called an astatic pair, and the instrument, an astatic galvanometer. It is not essential to place the coil in the magnetic meridian since the weak restoring field may be governed by control magnets. For ordinary use (small angles) the deflection is closely proportional to the current.

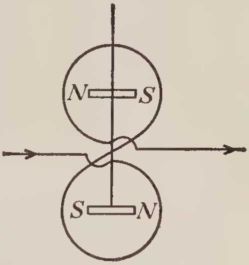


FIG. 204

340. Solenoids.—If the conducting wire of an electric circuit is wound closely and uniformly on a cylindrical frame, it forms what is called a solenoid. The meaning of solenoid is tube, and this winding forms in approximation a tube of conductor around which the current is caused to flow. Each turn of wire forms an equivalent magnet, and the solenoid may be thought of as a large number of such short magnets end to end, the north pole of one adjacent to the south pole of the next. Thus the external magnetic effect is the same as that of a cylindrical bar magnet. The lines of force due to a single turn no longer spread as soon as they emerge from its plane, but are confined to the interior of the solenoid except near the ends. Moreover, it may be shown by calculation that the magnitude of H is constant across the cross-section of the solenoid, and is given by the equation

$$H = 4\pi n I,$$

where n is the number of turns per cm length of the solenoid and I the current in emu. If I is expressed in amperes, the magnetic field in gauss is given by

$$H = \frac{4\pi nI}{10}.$$

It appears, therefore, that the intensity of field does not depend on the radius of the cross-section as it did for the single coil. This is a result of the superposition of the contributions of field from all the individual turns of wire, and is true only so long as the distance to the ends of the solenoid is great compared to the dimension of the cross-section.

341. Electromagnets.—If a cylinder of soft iron is introduced into a solenoid through which a current is flowing, it becomes powerfully magnetized. Poles are formed on its ends which greatly reinforce the external field which was originally due to the current alone, but which is now due both to the current and to the induced poles. This central cylinder of iron is called a core. The circuit and core form an electromagnet. The effect of the core on the electromagnet is to increase greatly its pole-strength and thus increase the value of H just outside. It becomes then a powerful agent for attracting and lifting masses of magnetic material.

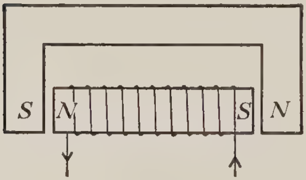


Fig. 205

The value of H in the interior of the iron is unfavorably affected by the strong poles at the ends of the core. Here, as always, H is due to the superposed effects of current and poles. Within the iron and near the north pole, the lines of force emerging from the pole will run in opposite direction to those of the current, and will thus tend to reduce the magnitude of the field below what it was before the core was introduced. It is the value of H within the iron, however, which determines the strength of the poles. Thus the poles in the core tend to demagnetize it. If the electromagnet is long compared with its cross-section, points within the iron near the middle are fairly free from these end effects. Here the value of H is nearly the same as before the iron was introduced.

If a yoke of soft iron is placed across the poles of the electromagnet as shown in Fig. 205, the induced poles in the yoke neutralize the effect of the core poles, the value of H rises within the core, and the strength of pole is considerably increased. Such an arrangement is called a closed magnetic circuit. In this case no poles produce any considerable external effects. The value of the field within the solenoid is again about $\frac{4\pi nI}{10}$ gauss.

342. Induced Electromotive Force.—In 1832, Faraday discovered that currents may be induced in closed circuits by moving magnets in their vicinity. This discovery directly and indirectly has been as fruitful as any in the history of physics. It forms the basis for modern dynamo-electric machinery.

Suppose a circular coil of wire is mounted vertically and forms a closed circuit through a galvanometer which will indicate both direction and magnitude of currents. If the north pole of a magnet is brought near the coil, a current will be seen to flow through the circuit and in such a direction that the circular coil presents a north pole to the oncoming pole of the magnet. If the magnet comes to rest, the current will cease to flow. If the magnet is moved from point A to point B, the strength of the current depends on the speed of the motion. If one measures the quantity of electricity transported, it can be shown that when the magnetic pole is moved from A to B the quantity is independent of the speed. The electricity transported in the circuit therefore seems to depend on the change in the magnetic field in which the coil finds itself, while the current which is the time rate of transport depends on the time rate of change of field.

If the north pole is now withdrawn from the vicinity of the coil, the current changes its direction, and we find that the coil presents a south pole to the receding north pole of the magnet. In both cases the polarity of the coil has been such as to oppose the motion of the magnet. This is an example of a more general principle called Lenz's law, which states that:

Whenever currents tend to flow in a circuit because of a change of magnetic field, the direction of flow will be such as to produce a magnetic field which opposes the change.

Thus, if the coil is originally in zero field and an endeavor is made to change it, currents will tend to flow so as to produce a field opposite to the impressed one. If we endeavor to reduce the value of an established field, the current will tend to maintain it. The effect of this will be that the agent is always required to do work in order to change the field at the circuit. The energy of the current induced is equal to the work done by the agent.

343. Other Methods of Inducing EMF.—If in place of a magnet, a second circuit carrying current is brought up to the fixed coil, a similar phenomenon takes place. So long as a current is flowing in the moving circuit, it has a magnetic field. It is, in fact, possible to think of it as equivalent to a magnet with definite polarity whose sign is dependent on the direction of flow of the current. If the north pole side of this circuit is brought near the fixed coil, a current will be induced of the same sign as that induced in the case described in the previous paragraph. Lenz's law tells us here also the direction of the induced current.

Moreover, if the second coil is also fixed but the current flowing through it altered, the magnetic field at the first coil will be changed and current again induced. If, for example, two coils are fixed parallel to each other, a growing current in one will induce a current of opposite sign in the other; a diminishing current in one will induce a current of the same sign in the other. This again is a consequence of Lenz's law.

We have said in the above that currents are induced by a changing magnetic field. This is a customary way of speaking, but the student is already aware that currents flow only as a result of an impressed potential difference or electric field or, if we consider the entire circuit, as a result of an electromotive force. Strictly speaking, the changing magnetic field has induced an EMF in the circuit and the current flows as a consequence if the resistance is not infinite. If, however, the circuit is open, the current will not flow, although the EMF is nevertheless induced. If also other EMF's are present in a series circuit, current will flow according to Ohm's law, $I = \frac{\Sigma E}{\Sigma R}$, where the induced EMF is included among the others with proper sign.

344. Magnetic Induction.—So far we have spoken of the direction of induced EMF and current and have noted in the cases taken that the magnitude of this EMF has depended on the rate of change of the magnetic field. Although this is often the case, it is not always true. The quantity H which we have called the intensity of the magnetic field and measured in gauss by means of the force on unit pole, is only one attribute of the field from the usual point of view. A second property is measured by the vector quantity B, which is called Magnetic Induction. As the name implies, it measures the ability of the magnetic field to induce EMF's in circuits which are brought into the region. Thus at every point in a magnetic field there is a value of H which determines the magnitude and direction of the force on unit pole if placed at this point, due to currents in the vicinity or induced poles or both. There is also a value of B which determines the magnitude of the EMF induced in a small coil, if rotated at this point about a certain axis in its plane with a certain speed. If the point is in free space, H and B are equal; if the point is in a magnetic material, they are not. To illustrate this, suppose we take a long solenoid and wind about it a few turns of wire which are connected to an instrument for measuring the flow of electricity. If a current is sent through the solenoid, a current will also flow through the second circuit when the solenoid circuit is being closed and also when it is being opened. We know that H in the solenoid depends only on the current and number of turns, not on the presence or absence of an iron core so long as we are far from the ends. The amount of electricity which flows through the second circuit, however, is much greater when the core is present. Although H is the same, we see B must be different when the iron is present.

345. Flux.—Flux is a word commonly used in electricity and magnetism, which has been borrowed from the mechanics of fluids. In the case of flowing fluids, we say that the flux of liquid through a certain area is the amount of liquid which flows through the area per second. If the velocity is at right angles to the area, the volume of liquid passing per second is Av, where A is the area and v the velocity of flow.

If the velocity makes an angle θ with the normal to the area, the volume traversing the area per second is found by multiplying A by the normal component of the velocity, or

$$\text{Flux} = Av \cos \theta.$$

This idea is carried over to other vector quantities even where no idea of flow is present. Thus we may speak of the flux of gravitational force through the floor of the room as FA, since the uniform force is everywhere normal to the horizontal floor. The flux of magnetic induction is found in a similar way. The flux of B is much more important to electrical engineers than the flux of any other vector quantity, and they are accustomed to use the word in this sense alone. We shall here follow this custom and the single word *flux* will always mean the flux of B. It will be indicated by the symbol Φ.

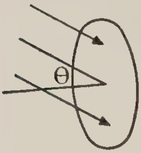


FIG. 206

346. Relation Between Φ and E.—The quantity Φ is defined with the help of the following equation,

$$E = N\frac{d\Phi}{dt}.$$

The sign is always given by Lenz's law. E is here measured in emu. N is the number of turns in the circuit through each of which the same flux Φ is changing. Since

$$1 \text{ esu of potential} = 300 \text{ volts} = 3 \times 10^{10} \text{ emu of potential},$$

or

$$1 \text{ volt} = 10^8 \text{ emu of potential},$$

we may write

$$E = N\frac{d\Phi}{dt}10^{-8},$$

where E is measured in volts.

If the magnetic field is uniform and perpendicular to the plane of the circuit,

$$\Phi = BA.$$

It is then possible to determine the value of B, or in any case its mean value from the value of Φ which in turn is determined from the induced EMF.

347. Permeability.—If B is measured with the help of the equation between E and Φ for regions in which H is known, it will be found that B is always equal to H *in vacuo.* B and H are related by the equation

$$B = \mu H,$$

where μ is called the permeability of the medium. Thus *in vacuo* $\mu = 1$. To determine μ, both B and H must be measured or calculated under conditions which will be described in §349.

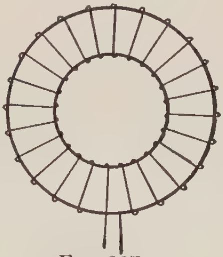


FIG. 207

For media both paramagnetic and diamagnetic which are only slightly magnetic, μ is very nearly characteristic of the material at a given temperature. For paramagnetic materials, μ is greater than one; for diamagnetic, μ is less than one. For ferromagnetic materials μ rises to large values and is far from constant. It varies with the magnitude of H.

348. Ring Solenoids.—A very convenient device used for the measurement of μ is a ring solenoid, shown in Fig. 207. This is a closed magnetic circuit. If the windings are close, it is free from poles and H is due to the current alone; it is independent of the core. If the cross-section of the solenoid is not too large as compared with its mean distance from the axis, H is nearly uniform within it, and is given by

$$H = \frac{4\pi nI}{10},$$

where I is the current in amperes and n is the number of turns per cm.

349. Magnetization Curves.—A few turns of wire passed around a ring solenoid form what we shall call a secondary circuit. Any alternation of current through the solenoid, or primary circuit, will cause current to flow through the secondary. If we begin with no current flowing through the primary and the core free from permanent magnetism, both B and H will be initially zero. A current I sent through the primary will increase H to $\frac{4\pi nI}{10}$. The number of turns in the secondary multiplied by the change of flux, $N_2\Delta\Phi$ is equal to $E\Delta t10^8$; or, since $E = RI$ where R is the resistance of the secondary,

$$E\Delta t = RI\Delta t = N_2\Delta\Phi\, 10^{-8};$$

but

$$I\Delta t = Q,$$

the amount of electricity flowing through the secondary which may be measured with a suitable galvanometer. We also know that if the flux is originally zero,

$$N_2\Delta\Phi = N_2(\Phi - o) = N_2BA,$$

where A is the cross-section of the solenoid; and, finally,

$$B = \frac{QR}{AN_2}10^8,$$

where Q is in coulombs, R in ohms and A in cms.^{2} It is customary to reverse the primary current during these measurements, starting with high values and decreasing by small steps. Under these conditions definite curves may be plotted of corresponding values of H and B which are characteristic of the material. Fig. 208 shows a number of these cúrves for different materials. μ may be found for any point of the curve by dividing the value of B by the corresponding value of H.

350. Hysteresis.—Reversing the current in the determination of the magnetization curves was equivalent to starting from the origin for the determination of each point of the curve. It gave the effect of imposing a given value of H on a piece of iron whose initial values of B and H were zero. This would be an unnecessary precaution if it were true that for a given iron the value of B depended on the value of H alone. It depends also on the history of the specimen; in other words, on the state of the specimen when the change of H is made. If, instead of reversing the current, it had been increased by successive small amounts till B reached a high value, then decreased, passing through zero and producing a large value of B in the opposite sense, the curve would be of the type shown in Fig. 209. Here it will be seen that as the current is reduced to zero ($H = 0$), B does not decrease much below its maximum value. It is necessary to reverse the field to a negative value Oa before B drops to zero. There is a lag between the value of H and the expected value of B. This phenomenon is more pronounced for hard steels than for soft iron. It is called hysteresis. It is usually an undesirable property in dynamo-electric machinery, since it is accompanied by a loss of energy in the form of heat. This loss may be shown to be propor-

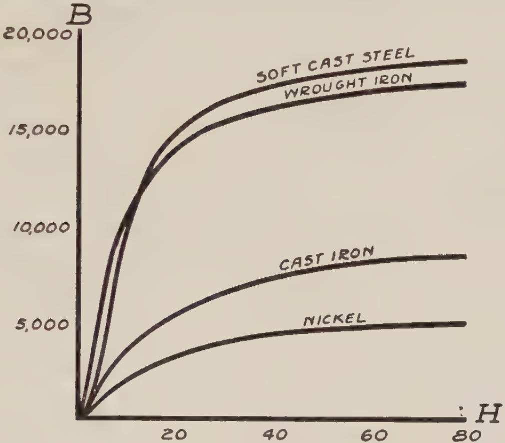


Fig. 208

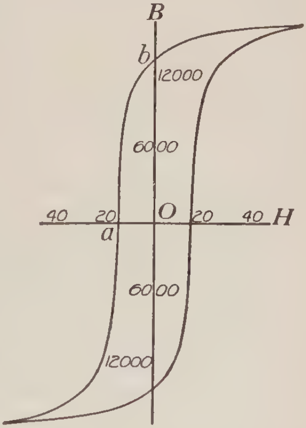


Fig. 209

tional to the area included in the hysteresis loop. *Ob* is the value of B retained in the iron if H is reduced from a high value to zero. It is called the remanence of the sample. *Oa* is the reverse field necessary to reduce B to zero. It is called the coercive force. The permeability of the sample cannot be calculated from this curve but must always be determined from the magnetization curve, as shown in §349.

351. Lines of Induction.—We have no such convenient name for the unit of B as we have for the unit of H. As is true for all vector fields, lines of induction may be drawn throughout a magnetic field which will indicate the direction and magnitude of B at every point. If the scale of spacing is one line per cm^2 for unit value of B, it is then possible to describe the value of B as a certain number of lines per cm^2. It must not be forgotten that the lines are only a means of graphical representation.

It should also be noted that in the determination of B in the experiments just described, it was really Φ which was measured directly. This has led the engineer to speak of B as "flux density" rather than magnetic induction.

352. Magnetic Circuit.—In closed magnetic circuits of constant cross-section, the values of both B and H are nearly uniform throughout. For example, in a ring solenoid closely and uniformly wound and containing a core of uniform permeability, this is very nearly true. The value of H everywhere within the solenoid is $\frac{4\pi nI}{10}$, and $B = \mu H$. For such a case, the equation between B and H may be thrown into a somewhat different form. If we call A the cross-section of the iron core and L its mean length, we may write

$$BA = \frac{A\mu HL}{L} = \frac{HL}{L/\mu A}.$$

BA is the flux through the core and equals Φ. HL is equal to the work done in carrying unit pole around the circuit and is called the "magneto-motive force" in analogy with electromotive force which is the work necessary to carry unit charge around an electric circuit. $\frac{L}{\mu A}$ is called the "reluctance" of the magnetic circuit. Writing with new symbols,

$$\Phi = \frac{MMF}{R}, \tag{1}$$

a relation similar to Ohm's law for electric circuits. It will be noticed that the expression for reluctance is similar to that for resistance. μ here replaces $\frac{1}{\rho}$, the electrical conductivity. Since

$$H = \frac{4\pi nI}{10},$$

$$HL = \frac{4\pi NI}{10},$$

where $N = nL$. N is the whole number of turns in the winding. The product NI is usually called the "ampere turns" of the circuit.

In practice, this equation is extended to magnetic circuits which are only approximately closed. We know that in the case of a straight solenoid, the values of B and H near the ends are far from being equal to their values near the middle. The lines of induction always form closed loops. None of them come to an end near the ends of the core, but some escape through the windings. This is called "magnetic leakage." The lines of H do this in part; some of them also end on the induced south pole of the core and start on the equal induced north pole. Since B and H are not uniform, equation (1) is not valid for this case. There is, in fact, no one value of Φ common to all parts of the winding. If the straight solenoid is bent into the form of a loop so that the ends approach each other, the demagnetizing effect of one end will be partly neutralized by the other. For a small air gap, B is again approximately uniform. The value of B in the air is nearly the same as in the iron, and we may write

$$\Phi = \frac{\dfrac{4\pi NI}{10}}{\dfrac{L_1}{\mu_1 A_1} + \dfrac{L_2}{\mu_2 A_2}},$$

where the subscripts 1 and 2 refer to air and iron respectively. Although only approximate, this formula is of great value to engineers in the designing of dynamo-electric machinery. It is possible to use it in cases where the circuit may be broken up into parts, in each of which B is homogeneous. Where slight leakage is present, a correction factor is applied, depending on the design.

353. Theory of Magnetism.—We have already learned that the strength of an electric current is the rate of transport of electricity. Wherever electricity is in motion, we shall find the characteristics of the electric current, and chief among these is its magnetic field. Rowland was able to show that an electric charge rotated on the circumference of a wheel shows magnetic effects. We know that the atom consists of a positive nucleus surrounded by a group of electrons revolving in orbits. We must expect, therefore, that many atoms will possess resultant magnetic moments. Knowing, for example, the orbital constants of the single electron of hydrogen, we might hope to calculate its average magnetic effects. An unmagnetized body may be thought of as one whose individual atoms possess zero magnetic moment because of the orientation of their various orbits, or as one in which the atoms, though magnetic individually, neutralize each other through their random orientation.

The effect of bringing an array of atoms into a magnetic field is twofold. One effect is to alter the motion of the electrons in their orbits, and the other is to turn the orbits into a new alignment. In the first place, the revolving electrons will behave as the current in the circuit toward which we brought a magnet. In obedience to Lenz's law, a rotating electron would tend to alter its motion so as to present a north pole to an oncoming north pole. Such an effect would cause the body to be repelled by the magnet, and we would describe it as diamagnetic. Superimposed on this is the other effect which in many cases overwhelms the first. Bringing the orbits into a magnetic field tends to turn them in such a way that they align themselves with the field. Where such an effect is predominant, the material is called paramagnetic. Ferromagnetic materials possess this quality very highly. Materials with remanence find difficulty in restoring the random orientation without the help of a reversed field. This theory also helps us to understand why it is never possible to isolate a magnetic pole. The two sides of these small circuits furnish the two poles of the magnet. Both must always be present and will furnish equal pole strengths.

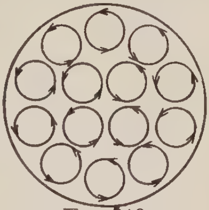


FIG. 210

354. Interpretation of B.—The two quantities H and B, which we have described as two attributes of the magnetic field, came into the subject long before a detailed theory of these phenomena was at hand. Though somewhat formidable to the beginner, they are of great value in engineering calculation and will doubtless never disappear from the subject, even though we now have in modern physics a somewhat different point of view. Suppose we take again the case of a long solenoid into which an iron core may be introduced. In the absence of the iron, we have noted that H and B are equal. When the iron is introduced into the magnetic field, the net effect is the addition of a large number of small currents, all flowing in the same sense, as shown in Fig. 210. If the field and medium are uniform, the adjacent parts of the small circuits neutralize each other, and the resultant current is entirely on the boundary of the iron. The effect on the secondary circuit is now due not only to the current in the solenoid but also to the "atomic currents" in the iron. The first of these effects is recorded by the quantity H, the two together by B. Thus B and H appear to have no essential difference but only to separate the current effects which may be controlled from those of the magnetic material. The permeability of a material tells to what extent this resultant current is called out in a given field, and it has a similarity to dielectric constant which records to what extent the atoms of a dielectric are polarized by a given electric field. It must be noticed that although the resultant current may flow in either direction (μ greater or less than one), the polarization of the dielectric is always with the electric field (k greater than one).

355. Mutual Inductance.—If two circuits are near enough so that each lies within the magnetic field of the other, any alteration of current in one will induce an EMF in the other. It is customary to speak of the one in which the current is controlled as the primary, and the other as the secondary. The EMF induced in the secondary by a given change of current in the primary will depend on the rate of change of the flux through the secondary due to the change of current in the primary. If no iron is present, this is proportional to the current in the primary, and we may write

$$N_2\Phi_{12} = MI_1,$$

where I_1 is the current in the primary and Φ_{12} the flux through the secondary due to I_1. N_2 is the number of turns in the secondary. The proportionality factor M will depend on tho shape of the primary and secondary, their relative position, and the permeability of the medium. It is independent of the current if μ is constant, and depends then only on the design of the circuits. M is called the mutual inductance of the circuits.

If I_1 is altered, we have

$$M\cdot\frac{dI_1}{dt} = N_2\cdot\frac{d\Phi_{12}}{dt} = E_2.$$

M is therefore numerically equal to the EMF induced in the secondary by a change of current in the primary at the rate of one unit per second. This definition holds also when iron is present. M is then no longer dependent on the design alone but varies with the current, since μ varies with H. If the EMF is measured in volts and the rate of change of current in amperes per second, the mutual inductance is given by the above equation in henrys. A pair of circuits has a mutual inductance of one *henry* if an EMF of one volt is induced in one circuit by a change of current at the rate of one ampere per second in the other. In order that the two circuits may have large mutual inductance, the medium should have high permeability and they should be so placed that the flux due to one passes entirely through the other. This is effectively achieved by winding both circuits about a single core of iron. Increasing the number of turns in the primary (N_1) increases the flux due to its current, and increasing the number of turns in the secondary increases the induced EMF. The mutual inductance is proportional to the product, N_1N_2.

356. Self-Inductance.—When the current is changing in a circuit, it induces EMF not only in neighboring circuits but in the original circuit as well. The starting current is accompanied by a growing flux through the circuit which, in accordance with Lenz's law, will induce a counter EMF, that is, an EMF opposite in direction to the growing current. This prevents the current from rising abruptly to its maximum value. Similarly, if the current is decreased, say by opening the circuit, an EMF is induced which tends to prolong it. If the rate of decrease of current is very great, as is the case on opening a switch, the induced EMF may easily be great enough to produce a spark. These self-induced EMF's obey the laws just discussed,

$$N\frac{d\Phi}{dt} = L\frac{dI}{dt} = E.$$

The factor L is called the self-inductance of the circuit. It depends, like M, on the shape of the circuit and the permeability of the medium. If two circuits are so placed that each encloses the entire flux of the other,

$$M = \sqrt{L_1 L_2}.$$

The unit of self-inductance is also the henry. The self-inductance of a circuit is one *henry* if an EMF of one volt is induced when the current is changing at the rate of one ampere per second.

357. Energy in the Magnetic Field.—When a steady current is flowing in a circuit, it is surrounded by a magnetic field. This magnetic field is the seat of energy; work must be done in order to establish it. In other words, work must be done in order to bring the current up to its final value. During the rise of the current, a counter EMF is induced. We have already learned in §285 that an EMF always accompanies any reversible transformation of energy into or out of the electrical form, and is a measure of this transformation. The induced EMF is a measure of just this sort of transformation, and the work done against it is equal to the magnetic energy produced. This may be calculated by multiplying each small amount of electricity transported by the accompanying counter EMF and summing these results from the time the current is started until it becomes constant and the counter EMF disappears. With the help of the calculus, this may be shown to give

$$W = \tfrac{1}{2} LI^2,$$

where L is the self-inductance and I the final current. If L is in henrys and I in amperes, W will be given in joules.

Whenever the current in a circuit is increased, work must be done to increase the energy of its magnetic field. This work is done against the counter EMF. Similarly, when a current is reduced, the energy of the field is again available and will "discharge" into the after current. The increasing current is thus retarded by the necessity of building up a magnetic field, and the decreasing current is prolonged by the release of this energy. Electricity appears to possess a property, analogous to inertia, which causes it to resist being set in motion or brought to rest.

358. Illustration of Self-Induction.—If a coil C of high inductance is connected in parallel with a lamp across the terminals of a battery with a resistance in series (Fig. 211), the phenomenon of induction may easily be seen. On closing the key, the lamp will flash up brightly but will soon after grow dim. The increasing current in C has encountered a counter EMF equal to $L\frac{dI}{dt}\cdot$ Immediately after the circuit is closed this counter EMF prevents any appreciable flow of current through the coil. As a result the lamp receives a potential difference which is a large part of the EMF of the battery. If the resistance of the lamp is R_L and that of the battery branch R_B, the initial potential difference over the lamp is

$$V_o = \frac{R_L}{R_L + R_B}E.$$

As the current rises to a steady value, the counter EMF disappears and the potential difference across the coil becomes

$$V = \frac{R}{R + R_B}E,$$

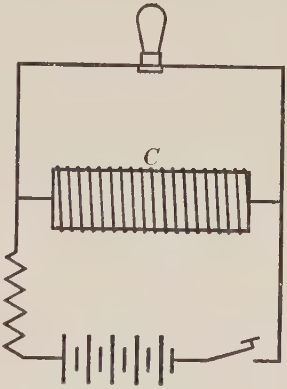


Fig. 211

where R is the resistance of the coil and lamp in parallel. The resistance of the coil is small, and therefore R is still smaller. V is consequently much smaller than V_o. When the switch is opened, the core of the electromagnet is demagnetized and the change of flux induces an EMF in the coil, which causes a current to flow through the lamp in a direction opposite to that produced by the battery. This induced current causes the lamp to flash as the switch is opened.

359. Induction Coil.—An induction coil is a device for transforming a low direct EMF into a much higher one in a second circuit. The primary circuit which contains the direct source, usually a battery, passes around the cylindrical iron core with a comparatively small number of turns. On the same core is wound the secondary with a large number of turns. (To avoid confusion, the secondary winding has been omitted from Fig. 212.) The primary circuit contains an interrupter *ab* which operates on the same principle as an electric bell. When the circuit is closed as shown in the figure, the core becomes magnetized and attracts the soft iron block which is attached to the spring *s*. The movement of *a* opens the circuit at the stationary metal contact *b*. The current then falls to zero, the core loses its magnetization and the block *a* is brought back to *b* by the spring. The primary circuit is thus provided with an automatic device for turning what would otherwise be a steady current from the battery into an interrupted current which is capable of inducing EMF in the secondary. The secondary EMF has the value $M\frac{dI_1}{dt}$. In order that $\frac{dI_1}{dt}$ may be high, the self-inductance of the primary is made low by reducing its number of turns. This also reduces M, but this is compensated by increasing the number of turns in the secondary to a very large number. We have already learned that M is proportional to N_1N_2.

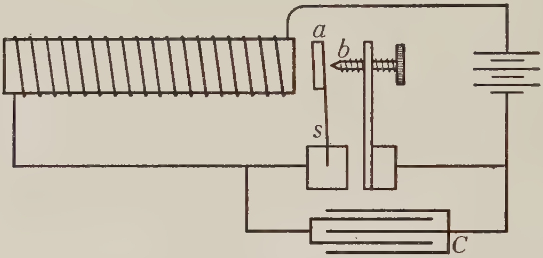


FIG. 212

As the primary circuit is broken, the value of H in the core falls to zero. It is essential that B should likewise fall to zero. This is secured to some extent by having the magnetic circuit open. The poles at the ends of the core succeed somewhat in demagnetizing it. If the core had formed part of a closed iron ring, the remanence would have prevented any large decrease in Φ without a reversal of current. A condenser C is connected across the interrupter to accomplish this still further. Now as the circuit is abruptly opened at ab, the consequent induced EMF charges the condenser to a potential difference much higher than the EMF of the battery. This immediately discharges, sending a current back through the battery and primary coil. Thus, the value of Φ is effectively reduced to zero or even slightly built up in the opposite sense, and a high secondary EMF is secured. Moreover, by providing a circuit for the after current from the coil, sparking at the interrupter and consequently fouling of contact are prevented. The abruptness of the current break, together with the action of the condenser, makes the resulting EMF in the secondary much greater than that which is induced when the primary circuit is closed. The secondary EMF is therefore almost unidirectional. It may easily be made several thousand times as great as that in the primary.

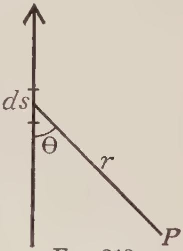


Fig. 213

This may be thought of as a direct current "transformer." These principles, together with the energy relations, will be discussed further in connection with the alternating current transformer.

360. Force on Current in Magnetic Field.—If a conductor carrying current is brought into a magnetic field, it is acted upon by forces which may tend to rotate it or even to displace it as a whole. Each element of the conductor ds may be thought of as acted on by a small force depending on its length and position. The circuit as a whole experiences the resultant of these small forces.

Suppose a positive pole of strength m is placed at a distance r from a current element ds, and that the line connecting element and pole makes an angle θ with the current element, as shown in Fig. 213. The force on the pole due to the current element is, from the law of Biot and Savart,

$$m\frac{dsI \sin \theta}{r^2}.$$

For the direction of current shown in the figure, this force is directed into the page at right angles to it, since this is the direction of the field due to the current at this point. By Newton's third law, the current element must experience a force equal and opposite to this, that is, an equal force again normal to the page but directed toward us. This is then the direction and magnitude of the force on the conductor element due to the field of the positive pole. The magnitude of the field due to the positive pole $+m$ at the element is $\frac{m}{r^2}$ and directed along the line r away from the pole. Calling this

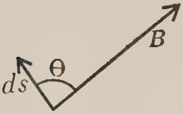


FIG. 214

$$H = \frac{m}{r^2},$$

$$F = \frac{mdsI \sin \theta}{r^2} = HdsI \sin \theta.$$

In case the medium has a permeability different from unity, the value of the field due to a pole m is

$$H = \frac{1}{\mu} \cdot \frac{m}{r^2}, \quad \text{and} \quad B = \mu H = \frac{m}{r^2}.$$

In general, therefore,

$$F = IdsB \sin \theta.$$

The direction of this force is at right angles both to ds and B. Its sense may be found by the following rule: If a right hand screw is turned in the direction $ds \rightarrow B$, it will progress in the direction of F.

Thus, in Fig. 214, where ds and B lie in the plane of the page, the rotation $ds \rightarrow B$ is clockwise. A right hand screw turned in this direction will progress into the page. When three vectors have the relation of *ds*, *B* and *F*, they are called a right hand set. It is clear that they must always be taken in the proper order.

This rule may be stated in another way particularly suited to its use in the important case of a motor. Fig. 215 shows two diagrams of lines of force, one in a uniform magnetic field and the other about a conductor carrying current normal to the plane of the paper and into the paper. If the conductor is brought into the uniform field, the two fields will be superposed and the lines will lie as shown in Fig. 216. Our rule tells us that the conductor experiences a force downward. This may be thought of as a consequence of a tendency of the lines of force which are crowded above the wire to straighten out and to repel each other laterally. The conductor will therefore move toward that side on which its own field is opposite in direction to that of the external field.

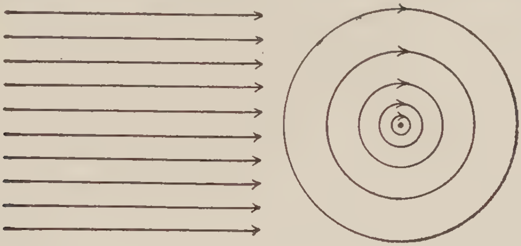


Fig. 215

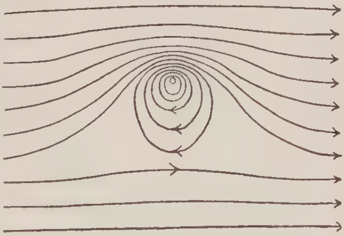


Fig. 216

361. D'Arsonval Galvanometer.—Fig. 169 showed the essential parts of the D'Arsonval galvanometer. A coil of wire carrying the current to be measured is suspended between the two poles of a magnet. It is possible to think of the coil carrying current as equivalent to a magnet with poles on its faces. Its tendency will then be to turn in such a way as to present its north pole to the south pole of the magnet, that is, to align its field with that of the magnet. Or it may be thought of in detail in terms of the formula derived in the preceding section. The two sides of the coil are acted on by forces equal and opposite in direction, which succeed in rotating it until counteracted by the torque of the twisted suspension. The coil stands in a magnetic field which is strong compared to that of the earth. It is therefore comparatively free from external magnetic disturbance. The galvanometer is the usual type of portable instrument and may be set up in any orientation. It is usually of moderate sensitiveness unless the number of turns is made very great and the resistance thereby greatly increased. An iron core is sometimes placed within the rotating coil to increase the value of B.

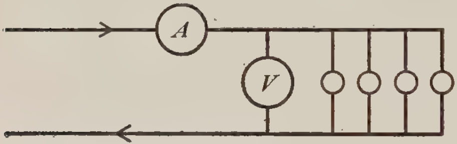


Fig. 217

The moment of inertia of the rotating coil may be considerable and the damping insufficient to bring it to rest as it returns to zero after the current is stopped. The return to zero may then be hastened by short-circuiting the coil. As it swings in the magnetic field on closed circuit, currents are induced which tend to stop the motion (Lenz's law) and transform the energy of its swing into the energy of the current.

362. Ammeter and Voltmeter.—The usual type of ammeter or voltmeter is essentially a D'Arsonval galvanometer whose coil is restored by a spring rather than by a twisted suspension. It is calibrated to read directly either the current which flows through it or that current multiplied by the resistance of the instrument, that is, its own fall of potential.

Since the ammeter is designed to measure current, it will always be inserted directly into that part of the circuit in which the value of the current is desired. It must therefore not alter the current by the introduction of a high resistance of its own. If the instrument is sensitive, this is usually accomplished by shunting its coil with a low resistance. The coil receives only a small part of the total current and is calibrated accordingly.

The resistance of the whole instrument is then less than that of the low resistance shunt.

A voltmeter, on the other hand, is not introduced directly into the circuit but connected in parallel with that part of the circuit of which the fall of potential is to be measured. It is necessary therefore that it shall not seriously alter the current in that part of the circuit. This is prevented by including in the case of the voltmeter a high resistance in series with the coil.

PROBLEMS

1. A single circular loop of wire 24 cms in diameter carries a current of 6 amperes. Find the intensity of magnetic field at the center.

2. The loop of problem 1 is placed with its plane in the plane of the earth's magnetic meridian. A small horizontal needle at the center of the loop is deflected by the action of the current 60° from the meridian. Find the horizontal intensity of the earth's field.

3. A long wooden cylinder has a coil of wire wound upon it. The coil has 2,400 turns and has a mean area of cross-section of 30 sq. cms. It has a length of 100 cms, which is great enough so that end effects may be neglected. When a current of 5 amperes is sent through the wire, what is the intensity of magnetic field within the helix?

4. In the preceding problem, find the magnetomotive force, the magnetic induction, and the magnetic flux.

5. An iron ring 30 sq. cms in cross-section has a mean circumference of 100 cms. It is wound with 2,400 turns of wire, and a current of 5 amperes is sent through the coil. The permeability of the iron is 1,200. Compare this with problem 3, then find the magnetomotive force, the magnetic induction, and the magnetic flux.

6. How would the results of the preceding problem be changed if a gap were made in the ring one cm across?

7. Compute the inductance of the coil of problem 5, and find the energy stored in the medium surrounding the coil when a current of 5 amperes is flowing.

8. A coil of 100 turns of wire is wound on a thin square frame, the mean area enclosed being 900 sq. cms. The coil is revolved in a horizontal field of uniform intensity of 80 gauss in air. It makes 600 revolutions per minute about an axis lying in the plane of the coil perpendicular to the field. (a) Find the average EMF developed. (b) Find the EMF when the coil is horizontal.

9. If a wire whose ends are connected to a galvanometer of 50 ohms resistance were to be put through the gap in the ring of problem 6, what quantity of electricity would flow through the galvanometer? Neglect the resistance of the wire.

10. An induction coil is so built that normal core flux is 25,000 lines of induction, and the condenser is of such a capacity that this flux can be taken out of the iron in .02 of a second. How many turns should the secondary coil have to give an average EMF of 2,000 volts at the break of the primary circuit?

CHAPTER XXV

CONDUCTION THROUGH GASES

363. Conduction Through Gases.—Dry air at atmospheric pressure is practically non-conducting for moderate difference of potential. If an electric field, not too intense, is applied between two electrodes separated by air, the few ions initially present are immediately swept out, and new ones do not form with sufficient speed to yield any appreciable current. The slight original ionization is probably due to external causes, chiefly short wave radiation which is always present. If the electric field is increased, a value will be reached at which the discharge takes place violently in the form of a spark. The initial ions are in this case accelerated so intensely by the high field that, even in the small time between collisions, velocities are attained which are great enough to produce new ions by impact.

This phenomenon may be studied more successfully by watching the discharge in a tube in which the gas pressure may be altered. Applying again a field too weak to cause spark discharge at atmospheric pressure, the current builds up rapidly as the gas pressure is reduced. A reduction of pressure has increased the length of path of ions between impacts. This permits them to attain high velocities in the electric field and to build new ions on collision. The effect is cumulative and the current may rise to several milliamperes. As the pressure is reduced still further, the chance of collision diminishes, the mechanism for producing new carriers gradually disappears and at still lower pressures the air becomes again non-conducting. There is an optimum pressure for a given potential difference and length of tube at which the current is a maximum.

364. Nature of Discharge.—As current starts in a discharge tube on reduction of pressure, the tube gradually fills with a bright glow, the color of which depends on the gas in the tube. Near the cathode is a characteristic dark space which extends farther into the tube as the pressure is reduced. It finally fills the tube and the glow disappears, although current is still carried between the electrodes. If the pressure is not too low the current through the tube is carried by a double procession of electrons and positive ions. The presence of these charges in varying proportions throughout the conducting gas alters decidedly the distribution of field. It has the effect of concentrating the major part of the total fall of potential between the electrodes into the region near the cathode. Electrons starting from the cathode find themselves immediately in this intense field, and they are powerfully accelerated until they experience their first collision. The average length of this free excursion is called the mean free path, and depends on the pressure. It is closely related to the length of the cathode dark space. Very intense ionization takes place near the beginning of the glow which is the most probable region of first collision. The products of the ionization migrate both ways in the field, the positive ion returning to the cathode where again intense ionization takes place. The bombardment of the cathode causes it also to emit electrons.

365. Cathode Rays.—During the dark low pressure discharge, ionization and recombination still take place, but too feebly to emit visible radiation. The mean free path of the electron is now as great as the dimensions of the tube itself. The most intense field still occurs near the cathode. Electrons starting from there receive almost their final velocities in a very small region and from there on traverse straight-line paths, bombarding the opposite side of the tube regardless of the position of the anode. This type of discharge was detected before electronic phenomena were widely known. The streaming electrons were called cathode rays. Very striking phenomena were found to accompany this discharge—for example, the luminescence of the glass tube at the point where the electrons arrive. The energy of each arriving electron is Ve, where V is the fall of potential and e the electronic charge. Since the mass of the electron is small, this means high velocities. Moreover, if the cathode is concave the stream may be focused on a small area. As a result, the molecules of the glass are excited and emit radiations characteristic of their composition. Some minerals exhibit this phenomenon very markedly. This should be distinguished from ordinary thermal incandescence which may also occur if the stream continues to bombard the same spot for some time. In fact, a metal target may be fused by continued bombardment. The arriving electrons also produce a marked mechanical effect. A vane may be made to rotate rapidly by this bombardment.

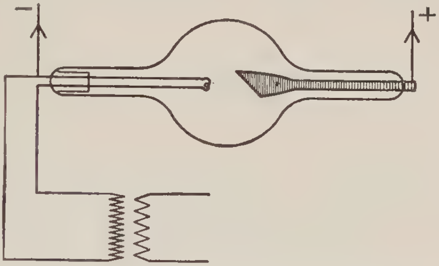


FIG. 218

366. X-Ray Tubes.—The X-ray tube does not differ essentially from the tube just described. Fig. 218 shows the arrangement of parts. The target, or anti-cathode, on which the electron stream is received is made of a material which has a high melting point, such as tungsten. The arriving electrons lose part of their energy in the form of radiation when they are brought to rest. The wave length of this radiation depends on the speed of the electrons, or, in other words, on the potential difference applied to the tube. As this speed goes up, the minimum wave length emitted becomes shorter. The wave length of ordinary X-rays is of the order of magnitude of $\frac{1}{1000}$ that of visible light. In addition to this general radiation, the tube also emits "bright lines" characteristic of the material of the target. Modern tubes are operated at extremely low pressures. This would normally reduce the current decidedly, but electrons are provided by a glowing cathode which is heated by a special circuit. Many materials emit electrons freely when incandescent and are used very generally as hot cathodes in vacuum tubes.

367. Motion of Electron in Electric and Magnetic Fields.—If a screen with a small aperture is placed in a cathode stream, it is possible to obtain a narrow beam which produces a small sharply marked spot on a fluorescent screen at the end of the tube. A strong electric field may be impressed at right angles to the path of the electrons by means of two charged plates, as shown in Fig. 219. The effect of the field will be to deflect the electronic beam and move the bright spot on the fluorescent screen. If the field is homogeneous throughout the path, it is easy to calculate the amount of the deflection. The component of the velocity normal to the field is constant, and equal to the original value v. If d is the length of path normal to the field, the time of transit is $\frac{d}{v}$. During this time the stream has been accelerated in the direction of the field. The acceleration is $\frac{Ee}{m}$, and the displacement will be

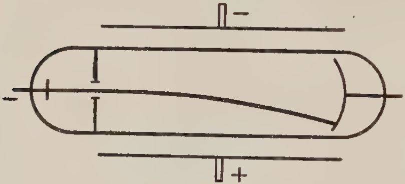


FIG. 219

$$s = \tfrac{1}{2}at^2 = \frac{1}{2} \cdot \frac{Ed^2}{mv^2},$$

where e is measured in esu.

Another way to deflect the electron stream is to cause it to pass through a magnetic field at right angles to its path. A moving electron has very similar properties to those of the current element Ids which has been used in the discussion of the forces on a conductor carrying current. In fact, the force on a moving charge is given by substituting ev for Ids in the previous formula. Therefore,

$$F = evH \sin \theta,$$

where e is measured in emu.

If an electron with velocity v is allowed to move in a uniform magnetic field at right angles to the lines of force, it will always experience the same force, evH. As the electron is deflected, the force will also turn so as to continue to be normal to the path. Since the force never has a component in the direction of the motion, it will never change the speed of the electron, and therefore evH will remain constant. A body which moves in obedience to such a force will execute a circular path, and the deflecting force will be equal to the centrifugal force of its motion.

$$evH = \frac{mv^2}{r}.$$

From this,

$$r = \frac{mv}{eH},$$

where r is the radius of the circular path.

If electrons of various speeds are projected through either a uniform electric or magnetic field, they will experience various deflections, depending on the velocity of projection. They are, in fact, spread out into a sort of velocity spectrum. The quantitative measurement of v will require a knowledge both of the magnitude of field, E or H, and of the ratio $\frac{e}{m}$, a characteristic quantity for the electron.

368. Measurement of $\frac{e}{m}$.—If a homogeneous field H is impressed at right angles to the page in Fig. 219, the electron stream will be deflected up or down in the plane of the page, depending on the direction of the field. The intensity of the magnetic field may be adjusted until the force due to it is just equal to that of the electric field discussed above. If then they are both applied at the same time and in opposite directions, the stream will be undeflected. Then

$$Hev = Ee\,3 \times 10^{10},$$

from which

$$v = \frac{E}{H} 3 \times 10^{10}.$$

It has been necessary to introduce the factor 3×10^{10} to convert the charge to a single system of units.

$$1 \text{ emu of charge} = 3 \times 10^{10} \text{ esu of charge.}$$

This determination of v depends only on the field intensities. If now the electron is deflected as described in §367, one may substitute this value of v in the previous formula and solve for e/m. The value of e/m was determined by this means before the value of e was known precisely. As a result of the experiment, we have for the electron

$$\frac{e}{m} = 1.76 \times 10^{7} \text{ emu per gram.}$$

Substituting $e = 1.59 \times 10^{-20}$ emu, we obtain

$$m = 9.0 \times 10^{-28} \text{ gram.}$$

The electrons are usually given their high velocities by allowing them to pass through a difference of potential V before entering the aperture. Then

$$Ve = \tfrac{1}{2}mv^2.$$

For $V = 1{,}000$ volts, $v = 1.87 \times 10^{9}$ cms per sec. This is nearly 12,000 miles per second.

CHAPTER XXVI

DYNAMO MACHINERY

369. Definitions.—When the principle of electromagnetic induction discovered by Faraday is applied in a device for converting mechanical energy into electrical energy, the machine whereby this is accomplished is called a generator. A similar machine for converting electrical energy into mechanical energy is called an electric motor.

The term dynamo machinery includes all machines used for either of these purposes. A dynamo machine consists of a system of conductors, called an armature, together with powerful electromagnets for producing the magnetic field. Either the armature or the field is mounted upon a shaft in order that it may rotate with respect to the other.

370. The Generator Principle.—We have expressed the magnitude of an induced EMF as the time rate at which the magnetic flux through a loop changes. When this change is accomplished by moving the loop in a magnetic field which itself is not varying, the rate of change of flux through the loop is also the rate at which the wire constituting the loop is cutting the magnetic flux; hence the induced EMF is equal to the time rate of cutting magnetic flux. The expression for EMF to which this statement leads is found convenient in dealing with dynamo machinery. In Fig. 220, lines of magnetic flux are represented by the arrows, and the line ab represents a wire at right angles to the field having a length l. B represents the flux density or number of lines per square cm. If the wire is made to move one cm in a direction perpendicular to the plane of the figure,

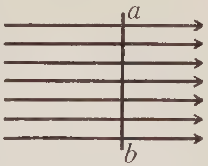


Fig. 220

it will cut Bl lines of induction; and if it moves with a velocity V, it will cut the flux at the rate of BlV lines per second; consequently it will have induced in it BlV emu of EMF. If the velocity V is not at right angles to the flux, its component in that direction is used in computing the EMF, and

$$e = BlV \sin \theta,$$

in which θ is the angle between the direction of the flux and the direction of the motion of the wire. To express e in volts, this value must be divided by 10^8.

371. Eddy Currents.—The generator principle finds application whenever there is a relative motion between a magnetic field and a conductor lying in it. If a conducting body is moved in a magnetic field, there will be induced in it currents which, by Lenz's law, are so directed that their magnetic fields oppose the motion. If the body is of large size, there are circuits in it of considerable extent so that these induced currents may be of large magnitude, and their effects are very apparent.

The following is a description of a simple experiment illustrating the production of these induced currents, which are called eddy currents. A circular disk of copper is arranged to swing as a pendulum between the poles of an electromagnet. It experiences little resistance to its motion so long as there is no current in the magnet coils. It swings freely and will vibrate for a long time; however, the moment the iron core is strongly magnetized, the pendulum is brought to rest. The induced currents flow around the disk in such a way as to produce poles which are repelled by the magnet as the disk approaches the magnet. The direction of the eddy currents may be predicted by applying Lenz's law. If a series of parallel saw cuts are made about two-thirds of the way through the disk, the circuits through which the eddy currents were flowing are broken, and the currents are confined to such narrow strips that they are practically eliminated. The disk will now swing between the poles of the magnet even when it is strongly magnetized.

In dynamo machinery, the armature winding is usually placed in slots cut in an iron core. If the iron were a solid casting, there would be eddy currents in it which would absorb much energy and greatly reduce the efficiency of the machine. To prevent this, the core is laminated, *i.e.*, it is built up of a large number of thin sheets of iron, insulated from each other either by varnish or by the thin layer of oxide that usually covers the iron. The construction is such that the direction of the induced EMF is perpendicular to the sheets so that there is no extended circuit in which the currents may flow. It is customary to use laminated cores in all cases where there is any considerable variation in magnetic flux.

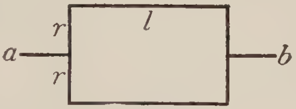


FIG. 221

372. The Alternating Current Generator.—Consider next a single rectangular loop of wire like that shown in Fig. 221. The loop is arranged to rotate in a uniform magnetic field about the axis ab, perpendicular to the lines of magnetic force. Fig. 222 shows an end view of the same loop, the axis in this figure being at right angles to the plane of the figure. V represents the velocity of that part of the wire which cuts the magnetic flux, and $V \sin \theta$ is the vertical component of this velocity. Since there are two parts of the wire, each of length l, cutting the flux with velocity $V \sin \theta$, the EMF induced is

$$e = 2BlV \sin \theta.$$

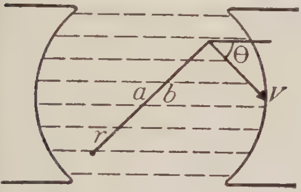


FIG. 222

When the plane of the loop is at right angles to the flux, $\sin \theta = O$ and $e = O$. When the plane of the loop is horizontal $\sin \theta = 1$ and $e = 2BlV$, *i.e.*, it has its maximum value. Since the EMF is a sine function of the time, it may be plotted as a sine curve. The current produced by such an EMF is evidently an alternating current. In order to utilize the current in an external circuit, two rings are fixed upon the shaft and insulated from it. The ends of the wire constituting the loop are fastened to the rings. A metallic or a carbon brush bears upon each ring, making a sliding contact with it as the loop rotates, and the external circuit is connected between the brushes. Fig. 223 shows an end view of the loop connected to the rings, which are shown of unequal sizes for the sake of clearness. In reality, the rings are of equal diameters and are placed side by side on the shaft.

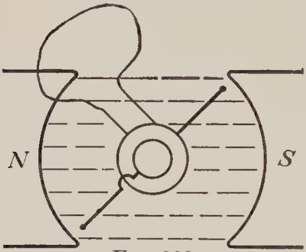


Fig. 223

The commercial generator is equally simple. For single phase currents, the armature circuit is made of conductors which cross adjacent pole pieces in opposite directions; and since adjacent poles are of opposite polarity, the induced EMF's are all in one direction in the circuit at any instant. When the conductors have moved along to the next pole, the EMF's are reversed. With a proper distribution of the winding, the EMF may be made to follow approximately the sine law. Sometimes the windings are pictured as if they were unrolled and spread out on a plane surface; it then becomes easy to trace the circuit. Fig. 224 shows such a diagram, which for simplicity is drawn with only one conductor across each pole face. The two ends of the armature circuit are connected to two collector rings.

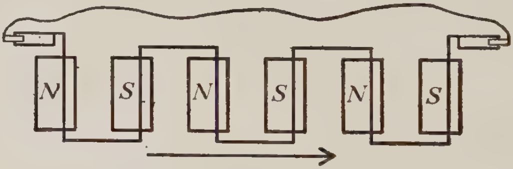


Fig. 224

With large machines the usual practice is to make the armature stationary and have the field magnets revolve. The slip rings are then connected to the field winding instead of to the armature winding.

373. The Direct Current Generator.—Again consider a single rectangular loop of wire rotating in a horizontal magnetic field, as in Fig. 225. The ends of the loop are connected to the two halves of a split cylinder upon which brushes bear for taking the current from the moving coil to the stationary part of the circuit. When the plane of the coil is vertical, the EMF passes through its zero value, and the brushes are so placed that the connection to the external circuit is reversed at the instant the current in the coil reverses. In this way the current in the external circuit is made to flow always in one direction, although it fluctuates in magnitude from zero to a maximum, as is shown in Fig. 226. The split cylinder is called a commutator, as is also the corresponding part of a more complicated machine.

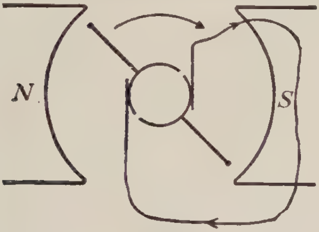


FIG. 225

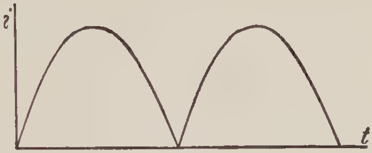


FIG. 226

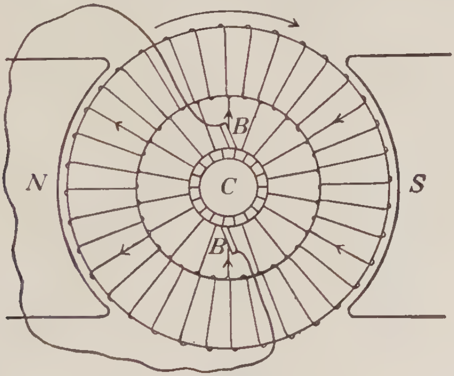


FIG. 227

374. The Ring Armature.—Fig. 227 is a diagram of a two-pole generator having a ring armature. A ring of laminated iron is wound uniformly with wire and the ends of the wire are joined together to form a closed circuit. At equal intervals along the winding, wires are connected which go to segments on the commutator *C*. The commutator is made up of a large number of segments which are insulated from each other by mica. The magnetic flux from the north pole enters the iron core of the armature and divides into two parts. One part passes through the upper half of the ring and the other through the lower half. Both parts follow the iron around and pass through the air gap at the right and then into the south pole. The EMF's developed in all the turns in the left half of the armature are in one direction, and their sum represents the EMF of the machine. In the right half an equal EMF is developed. The two halves of the armature are in parallel so that the EMF of the machine is one-half the sum of those developed in the several turns on the armature. The brushes bear at the top and bottom of the commutator. They are usually given a slight forward lead in the direction of rotation because the magnetic field is distorted by the reaction of the field, due to the armature current. By applying Lenz's law and finding the directions of all the EMF's in the different conductors, it is easily seen that the lower brush is positive, *i.e.*, the current leaves the armature by this brush and enters it by the upper brush.

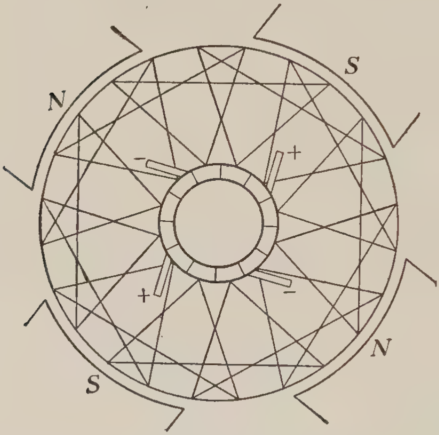


FIG. 228

375. The Drum Armature.—A four-path armature for a four-pole machine is shown in Fig. 228. The wire is wound in slots cut parallel to the axis of a cylinder of laminated iron, the slots being perpendicular to the plane of the figure. The chords of the circle represent wires behind the armature, and the lines running to the commutator represent wires in front of the armature. There are half as many commutator segments as there are slots for the windings, and the span of each coil is one less space than a quarter of the total number. The two positive brushes are connected together, as are also the two negative brushes. The EMF is a quarter of the sum of the EMF's in all the conductors. Since there are four parallel paths for the current through the armature, the internal resistance is 1/16 of the sum of the resistances of all the conductors, and the current carrying capacity is four times that of a single conductor. To compute the EMF of a generator, we make use of the fundamental principle that an induced EMF is equal to the time rate of cutting magnetic flux. The student would find it profitable to develop a formula for the EMF of a machine such as that shown in Fig. 228.

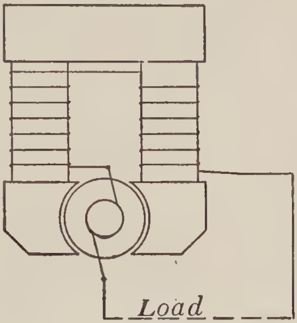


FIG. 229

376. Field Excitation.—In direct current generators the magnetic field is furnished by large electromagnets excited by current from the armature. The residual magnetism in the iron cores is sufficient to produce a small current in the revolving armature, which, when passing through the coils, increases the magnetism in the iron. As a result of this increase, a larger current is produced in the armature. This, in turn passing through the field winding, still further increases the strength of the magnets. Thus the EMF and the field strength are built up together until the machine is furnishing its full voltage. The series winding, shown in Fig. 229, is used when a constant current is needed, such as for arc lighting. In this case the entire load current passes through the field coils. The other simple type of winding, shown in Fig. 230, is called the shunt winding. The field coils consist of many turns of fine wire and are connected directly across the

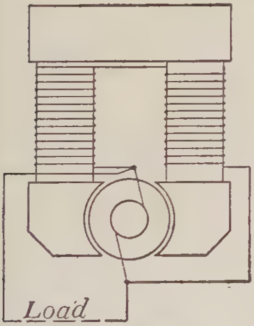


FIG. 230

brushes. This machine develops its maximum voltage when there is no load current. As the load comes on and the current in the load circuit increases, the potential at the brushes falls off because of the *IR* drop in the armature. This results in a decreased field excitation, and thus in a further reduction in the terminal potential difference.

The ideal machine would keep its terminal potential difference constant for all loads. This is practically accomplished by combining the series and the shunt method of exciting the field magnets. The shunt winding is like that of Fig. 230, but the load current is made to pass through a few turns wound on the cores of the field magnets. When the load current is zero, these series turns have no effect; but as the load current increases, they add their effect to that of the shunt coils, and this increases the strength of the field magnets. In this way there may be complete compensation for all tendencies to lower the voltage. Machines with this double field winding are called compound machines.

377. The Direct Current (D. C.) Motor.—If a current from some external source were to pass through either of the armatures shown in Figs. 227 or 228, magnetic forces would be produced which would tend to turn the armature on its axis. Assume that each conductor has a length l and carries a current I, and that the flux density in the air gap between armature and field pole piece is B. From §360, a conductor that lies in the magnetic field and perpendicular to it experiences a force equal to $\frac{BIl}{10}$ dynes at right angles to the field, provided I is measured in amperes. This force, being approximately perpendicular to the radius of the cylindrical armature, exerts a torque upon it. The sum of these torques produces the rotation, and in this way electrical energy is converted into mechanical energy. In Fig. 231, A and B are two conductors on opposite sides of the armature shown in cross-section. The current in A is from front to back in the picture, and that in B is in the opposite direction. The magnetic field shown by the iron filings is the resultant of the field due to the magnetic poles, N and S, and that due to the current in the conductors. It is evident that the direction of the torque is such as to produce counter clockwise rotation. The same figure would serve to illustrate the resultant magnetic field in the case of a generator whose armature has a clockwise rotation. One may get from the figure a suggestion as to why the brushes of a generator are rocked forward and those of a motor are rocked backward. The field is distorted so that it is not horizontal as a result of the armature reaction. The brushes are set on the neutral line which is through the center of the armature perpendicular to the resultant field.

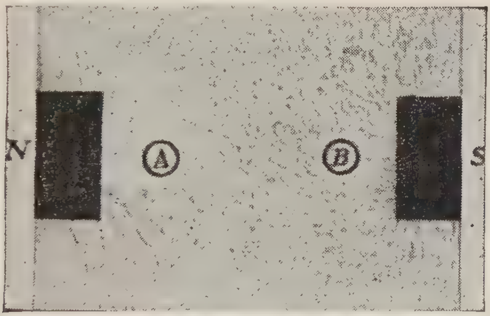


FIG. 231

Any generator may be used as a motor. When a shunt machine is used, the direction of rotation is the same whether the machine is acting as a generator or a motor; whereas, for a series machine, the direction of rotation as a motor is opposite to that as a generator. The direction of rotation of either the shunt or the series motor is not changed if the direction of the current is reversed. The student should think over these three statements and satisfy himself that he knows why they are true.

378. Counter EMF in a Motor.—When the armature of a motor is rotating, its conductors are cutting magnetic flux and there is induced in it an EMF which depends upon its speed just as in the case of a generator. This induced EMF is in a direction contrary to that of the current, and hence is called a counter EMF. It is this counter EMF that limits the current to small values when the motor is carrying little or no load. If more load is put upon the motor, its speed is slightly reduced, the counter EMF is reduced, and more current is admitted to the armature. Thus it is seen that the motor takes energy from the source of supply when a load is put upon it, and that it automatically reduces the current used in operating it when the demand for mechanical energy is decreased. If E is the constant potential difference applied to the motor, R the armature resistance, I the armature current, and E' the counter EMF,

$$E = E' + RI.$$

If E' is decreased as a result of a slight reduction of speed, I must increase since E is constant. Multiplying both members of the equation by I, we have

$$EI = E'I + RI^2.$$

EI is the applied power, I^2R is the power lost in producing heat in the conductors, and $E'I$ is the electrical power converted to mechanical power. For a good motor the last term in the equation is small so that the converted power is a large fraction of the applied power.

379. The Starting Box.—When the motor is at rest and is to be started, the line voltage should not be applied directly to the armature, since E' is zero and the current would be E/R. Since R is very small, this current would be sufficient to blow a fuse if it did no more serious damage. A starting box must be used in circuit with the armature, which furnishes sufficient resistance to cut down the current to a safe value. As the motor gains in speed, the resistance in the starting box is gradually reduced until, at full speed, it is made equal to zero. The controller used in starting a street car is a familiar illustration of a starting box.

380. The Two-Phase Generator.—The difficulty of operating large motors on single-phase alternating currents has led to the introduction of two-phase and three-phase systems. Fig. 224

illustrates the armature winding for a single-phase generator. If a second winding is placed upon the armature with conductors midway between those of the first winding, as in Fig. 232, we shall have a two-phase generator. There are four collector rings instead of two, and usually the external circuits are entirely separate. With similar external circuits, the two currents differ in phase by 90°. The curves of Fig. 233 show how the two currents are related. The reason for adding this complication to the A. C. generator will appear in the next sections.

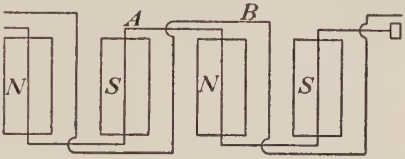


FIG. 232

381. The Rotating Field.—Suppose that we have a ring of laminated iron with four coils wound upon it, as shown in Fig. 234. Designate one pair of coils as the *A* winding, and observe that these coils are so connected that they produce a magnetic flux from left to right in both the upper half and the lower half of the iron ring when the current enters at *A* and leaves the coils at *A′*. The flux emerges from the iron at the right side of the ring, and a large part of it crosses the space in the interior of the ring and enters the iron at the left side. Thus there is a north pole at the right side and a south pole at the left side of the ring. If the current were to enter the other winding at *B* and leave it at *B′*, the north pole would be at the top and the south pole at the bottom of the ring. Suppose now that we connect these two windings to the brushes

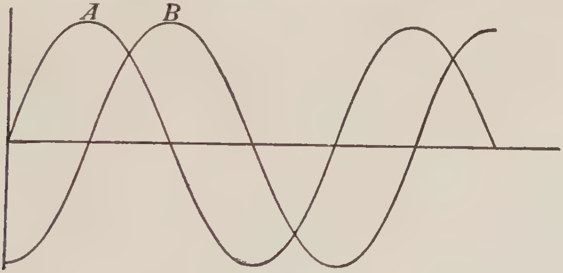


FIG. 233

of a two-phase generator. When the current in the A winding is at its maximum, that in the B winding will be zero. The poles will then be at the right and left sides of the ring respectively. A quarter of a cycle later the current in the A winding will be zero and that in the B winding a maximum; thus the poles will have shifted to the top and bottom of the ring. The poles actually move along the stationary ring as the two currents change in magnitude. After another quarter cycle, the current in the A winding will be at its maximum in the reverse direction and the poles will be shifted 90° farther on. Thus it is seen that we have a rotating magnetic field, the poles making as many revolutions per second as there are complete cycles per second in the alternating current.

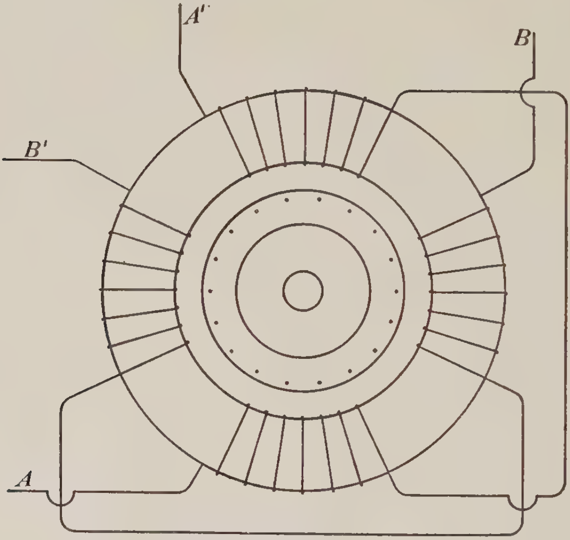


Fig. 234

382. The Induction Motor.—If the ring of Fig. 234 is laid in a horizontal position and a magnetic needle is poised at the center of the ring, the needle will tend to follow the rotating poles and will be set in rotation around its axis. A copper cylinder supported upon a vertical axis near the center of the ring will have eddy currents induced in it and will be set in rotation by the rotating field. In the figure there is shown an armature made by inserting copper rods in holes in a laminated iron ring. The copper rods are parallel to the axis of the ring and are electrically connected to end plates or rings also made of copper. Thus there are formed in the armature many closed circuits of very low resistance. This is called a "squirrel cage armature" because of a fancied resemblance to a cage that was not uncommon a generation ago. When the field rotates, there is a cutting of lines of magnetic induction by the conductors in the armature, and large currents are induced in them. We then have a conductor carrying a current at right angles to a magnetic field, and hence experiencing a force equal to $\frac{BIl}{10}$ dynes. The sum of the torques due to such forces as this will cause the armature to rotate, the direction of rotation being that of the rotation of the field.

The speed of the rotating armature or rotor is necessarily less than that of the rotating field, for if it were equal to that of the field there would be no induced current and the torque would be zero. When the machine is running without load the difference in these two speeds is small and the induced current is small. With increased load the speed of the rotor is slightly reduced, more current is induced and a larger torque is produced. The machine is therefore self-regulating, automatically drawing more energy from the source of supply when the demand is increased. It is to be observed that there are no brushes or other sliding contacts in this motor. If the bearings are properly lubricated, it requires no further attention. The two-phase motor may be made highly efficient and is, on the whole, a very satisfactory machine.

The three-phase system is to be preferred to the two-phase system and is at present more widely used. The principle underlying its operation differs so little from that of the two-phase system just described that it will not be necessary to discuss it here.

PROBLEMS

1. A two-pole generator having pole faces of area 240 sq. cms each, has a magnetic induction of 10,000 lines per sq. cm in the air gap. There are 260 conductors on the armature in the form of two parallel circuits. The speed is 1,000 R.P.M.; find the EMF developed.

2. A four-pole generator has a Gramme ring for an armature. There are four brushes and four parallel paths for the current in the armature. The total flux from each pole is 2,500,000 lines of induction. There are 600 turns of wire on the ring. At what speed is it running, if it develops 220 volts?

3. A 550-volt generator delivers 165 kw. to a pair of transmission lines whose combined resistance is .15 ohm. The power is delivered to motors at the end of the lines. What power is delivered, and what is the efficiency of transmission?

4. A series generator for arc lighting has a resistance of 15 ohms. It delivers 6.6 amperes and has a terminal potential difference of 2,400 volts. What is the EMF induced in the armature? If the mechanical and magnetic losses amount to 800 watts, what h.p. will be required to run it, and what is its efficiency?

5. A shunt motor operates on a 220-volt circuit and runs at 1,200 R.P.M. It has an armature resistance of 1.2 ohms and a field resistance of 550 ohms. The mechanical and magnetic losses are 180 watts. Full load current is 10 amperes. Find the full load efficiency.

6. Find the full load torque in the preceding problem, using metric units.

7. Each pole face of a four-pole motor is 16 cms square. The flux density in the air gap is 9,000 lines of induction per sq. cm. There are 96 conductors in front of each pole, each carrying a current of 8 amperes. Determine the total tangential force on the armature and the torque, if the mean radius of the winding is 12 cms.

8. Determine the counter EMF developed in the motor of problem 5.

CHAPTER XXVII

ALTERNATING CURRENTS

383. Introduction.—As explained in §372, an alternating current is one which periodically changes direction, rising to a maximum in one direction, falling to zero and then rising to a maximum in the reverse direction. When we plot the current as ordinate and the time as abscissa, as in Fig. 235, the curve is not usually a perfect sine curve; however, it is customary to develop the theory of alternating currents on the assumption that both EMF and current are harmonic functions of the time, *i.e.*, that the curve is a true sine curve. In connection with the study of simple harmonic motion, we found that the ordinate of a sine curve is proportional to the sine of a varying angle whose value at any instant is ωt, ω being the angular velocity in the circle of reference, and equal to $2\pi/T$ or $2\pi n$, where n is the number of revolutions per second. Each revolution in the circle of reference corresponds to a complete cycle of the wave; hence n is also the number of cycles per second, so that a 60-cycle current is one for which there are 120 reversals of the current per second. Using ω to represent $2\pi n$ and I to represent the maximum value of the current, the instantaneous value is

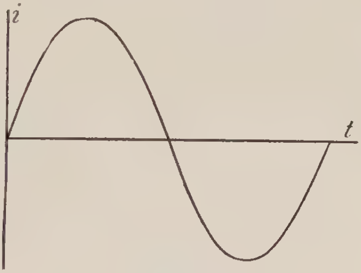


FIG. 235

$$i = I \sin \omega t.$$

t represents the time, counting from some instant when the current passed through its zero value and was increasing in a direction which is arbitrarily chosen as positive. The instantaneous value of the EMF is represented by a similar equation except that e and i may be in different phases, as will appear later.

384. Energy Distribution.—Wherever electrical energy is to be distributed over considerable territory, alternating currents are almost invariably used. Energy for city street cars constitutes the notable exception. To appreciate why the alternating current is to be preferred to the direct current, we must consider first the conditions for the economical distribution of energy. With direct currents the energy conveyed by a pair of wires is equal to the product EIt, E being the potential difference between the wires, I the current in each wire, and t the time during which the current flows. From this equation it is apparent that for a given rate of transmission of energy, I may be decreased if E is correspondingly increased. The fall of potential along a wire equals IR; hence the current should be kept small in order to have the line drop small. Furthermore, the power loss in the wire is I^2R, and this quantity should be kept small for good efficiency. The lamps used for house lighting require 110 volts and take a current proportional to the number of lamps operated. If the current could be transmitted from the lighting station at 2,200 volts, only 1/20 as much current would be transmitted as if the wires in the streets had a potential difference of 110 volts; and the energy loss for a given wire would be only 1/400 as great. This is precisely what is done in most cities, and it is made possible by the alternating current transformer which receives energy at 2,200 volts and delivers approximately 20 times as much current at 110 volts. Since a direct current can be efficiently transformed to a current of different voltage only by running machinery requiring an attendant to operate it, the advantage of using alternating currents becomes apparent.

385. The Transformer.—It is somewhat remarkable that two coils of wire wound on an iron core can perform such a valuable service and do it with so little loss of energy. The efficiency of large transformers is usually more than 97 per cent. A real appreciation of the behavior of a transformer can hardly be gained at this stage of our study, but the fundamental process of its operation should be considered now. Two coils of wire marked *P* and *S* in Fig. 236 are wound on a laminated iron core. An alternating voltage is applied to the primary winding *P*, which causes a small magnetizing current to flow in the coil. The magnetic flux thus produced is alternating, and hence there is induced in the secondary coil *S* an alternating EMF which at every instant is equal to the number of turns in the coil multiplied by the time rate of change of the magnetic flux in the iron core. In the ideal transformer we assume that the coils are without resistance; hence the induced EMF in the primary must be equal and opposite to the applied EMF, since there is assumed to be no *IR* drop in the coil. Since the same magnetic flux threads through both coils, the two EMF's induced are in the ratio of the numbers of turns in the two coils. If the secondary has more turns than the primary, the transformer is called a step-up transformer, whereas in the step-down transformer the number of turns in the primary is greater. The ratios most frequently used are 10 : 1 and 20 : 1. With such a transformer the current for lighting is transmitted at 2,200 volts and transformed, where it is to be used, to 110 volts.

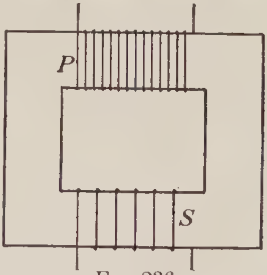


Fig. 236

When there is no load on the secondary of the transformer, only enough current flows in the primary to magnetize the iron, and very little power is used. This is why the transformer may be left connected to the lines continuously without much loss of energy. When the secondary is supplying current for electric lighting or for other purposes, this current reacts upon the iron core in such a way as to oppose the magnetizing action of the current in the primary. Thus there is a tendency to decrease the induced EMF in the primary below the value of the applied EMF, and hence to admit more current into the primary coil. This additional current in the primary, which flows only when the transformer is furnishing current in the secondary circuit, is called the load current. The magnetizing action of the load current must be sufficient to counteract the demagnetizing action of the secondary current; hence the following equation is true:

$$N'I' = N''I'', \text{ or } \frac{I'}{I''} = \frac{N''}{N'}.$$

From this we see that the currents are inversely proportional to the numbers of turns in the two coils. If the secondary circuit is open $I'' = 0$, and hence the load current $I' = 0$. In this discussion we have attempted to show how the transformer automatically adjusts itself to the demands that are made upon it, taking only a very small current to magnetize the iron and, in addition to that, drawing energy from the lines only when it is called upon to deliver energy for some useful purpose.

386. Electromagnetic Repulsion.—In §335 it was explained that parallel wires carrying currents in opposite directions repel each other, and in the preceding topic it is stated that the primary and secondary currents in a transformer produce opposing magnetomotive forces, *i.e.*, that I' and I'' are opposite in direction. Since this is the case, the primary and secondary coils should repel each other.

The statement that the primary current is opposite in phase to the secondary current is true only of the component of the primary current known as the load current. When an open magnetic circuit is employed, the magnetizing current is necessarily much larger than when the magnetic circuit is closed, and it may be a considerable fraction of the total primary current. If we keep in mind, however, that the primary current has one component, the load current, which is opposite in phase to the secondary current, the following experiment illustrating electromagnetic repulsion will be understood. A large bundle of soft iron wire, shown in Fig. 237, constitutes a core upon which is wound a coil of insulated copper wire through which a 60-cycle alternating current will flow when the switch is closed. A heavy copper ring is placed over the coil and supported near its upper end. When the switch is closed the ring is repelled and thrown upward away from the electromagnet. An ammeter in the primary circuit shows a much larger current when the ring is held in a position surrounding the core than when it is taken away. The added current corresponds to the load current in the transformer. This same apparatus is used for other experiments illustrating the action of a transformer.

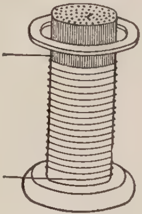


Fig. 237

387. The Constant Current Transformer.—Electromagnetic repulsion, which was explained in the preceding topic, finds an important application in the constant current transformer for operating street lamps in series. Fig. 238 is a diagram of such a transformer. The secondary coil, which furnishes the constant current, is supported above the primary coil and is free to move up or down if the force supporting it should vary slightly. The magnetic flux produced by the primary current meets an opposing magnetomotive force at the secondary winding, and hence some of the flux leaks across from the middle portion of the iron core to the outside branches without going through the secondary coil. The result of this leakage is that the EMF developed in the secondary winding becomes less as the coils are moved apart. If some change occurs in the external circuit to reduce its resistance, as when a lamp goes out and its place is taken by a short circuit, the current is slightly increased, the repulsion between the coils becomes greater, and the secondary is lifted to a posi-

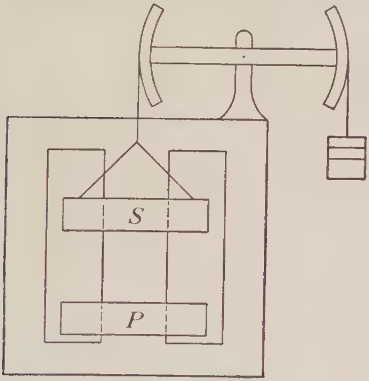


Fig. 238

tion such that the leakage is increased and the EMF is reduced. This serves to maintain the current at very nearly a constant value.

388. Measuring Instruments.—If an alternating current is passed through an instrument in which either the moving or the stationary part is a permanent magnet, as in the case of the ordinary galvanometer, the torque brought to bear upon the movable system will be alternating and, in most cases, the instrument will show no deflection. Instruments for alternating current measurements that make use of magnetic forces to produce a definite deflection must be so designed that the polarity of both the moving and the stationary parts is reversed at the same frequency. Such an instrument is said to be of the dynamometer type. It has two coils usually connected in series, one of which is movable and the other stationary. One produces a magnetic induction B in which the other moves when a current is flowing through it. The force tending to move the coil is BIl, but since B itself is proportional to I which produced it, the force is proportional to I^2. When an alternating current flows in the coils, the torque tending to move the coil is always in one direction, and its average value is proportional to the average of I^2. To determine the current indicated by such an instrument, we must take the square root of its angular deflection. Let α be the angular deflection, and k^2 the proportionality factor; then

$$\text{Average } i^2 = k^2\alpha,$$

and

$$\sqrt{\text{Average } i^2} = k\sqrt{\alpha}.$$

Considering k and α as known quantities, we have a measure of the square root of the mean square of the current. It is this value of the current, rather than the average numerical value, that all alternating current instruments measure. Furthermore, this is the value we should prefer to have measured, since energy equals the average of $i^2Rt = Rt$ Average i^2, and that quantity which when squared gives Average i^2 is $\sqrt{\text{Average } i^2}$.

Alternating current instruments usually have scales of unequal parts, so made that the numbers of the scale represent the square roots of the actual angular deflections. In this way the instrument is made direct reading.

389. Mean Value of Harmonic EMF.—Of course the mean value of any varying quantity that can be represented by the ordinates of a sine curve is zero, if the average is taken over an integral number of complete periods. We shall use the term mean value to signify the mean of the numerical values without regard to sign. Imagine a uniform field and a rectangular loop of wire revolving in it. The dimensions of the loop shown in Fig. 239 are $2r$ and L, L being the dimension perpendicular to the plane of the figure. Let e be the instantaneous value of the EMF, and V the velocity of the part of the wire that cuts the magnetic flux; n is the number of revolutions per second. Then $e = 2\,BLV \sin \theta$, as shown in §372. Since the maximum value of the sine is 1, the maximum value of the EMF is

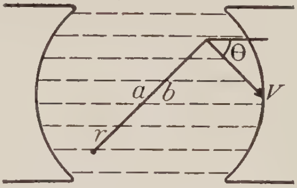


FIG. 239

$$E = 2\,BLV = 2BL\,2\pi rn.$$

To obtain the expression for the mean value of the EMF, we find the total flux cut in a second as follows: when the loop is in a vertical plane the flux passing through it is a maximum and equals $2rLB$. While the loop turns through 90° the flux is reduced to zero. The change in flux per revolution is, therefore, four times the maximum flux, or $8rLB$; and since there are n revolutions per second, the flux cut per second is $8rLBn$. Since the induced EMF is the time rate of cutting flux,

$$\text{Mean } e = 8\,rLBn \text{ emu.}$$

The ratio of the mean value of the EMF to the maximum value is

$$\frac{\text{Mean } e}{E} = \frac{8\,rLBn}{4\,rLBn\pi} = \frac{2}{\pi};$$

$$\text{Mean } e = \frac{2\,E}{\pi} = .636\,E.$$

390. Square Root of Mean Square.—Again we shall represent instantaneous values by e, maximum values by E, and the effective or square root of mean square value by E'. We are interested now in finding the "root mean square" value (R.M.S. value) in terms of the maximum value E.

$$e = E \sin \omega t, \quad \text{and} \quad e^2 = E^2 \sin^2 \omega t;$$
$$\text{Average } e^2 = E^2 \text{ Average } \sin^2 \omega t;$$
$$\text{Average } \sin^2 \omega t + \text{Average } \cos^2 \omega t = 1.$$

In an integral number of cycles, the average of $\sin^2 \omega t$ equals the average of $\cos^2 \omega t$, from which it follows that

$$2 \text{ Average } \sin^2 \omega t = 1,$$

and

$$\sqrt{\text{Average } \sin^2 \omega t} = \sqrt{1/2};$$
$$E' = \sqrt{\text{Average } e^2} = E\sqrt{1/2} = .707\ E.$$

It thus appears that the R.M.S. value of the EMF is 11 per cent greater than the average value if the EMF follows the sine law.

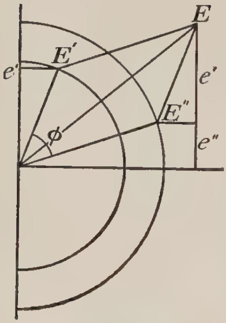


FIG. 240

391. The Circle Diagram.—When two alternating EMF's are applied to the same circuit, their instantaneous values may be added algebraically. However, their maximum values and their R.M.S. values can be added in this way only if they are in the same phase or in opposite phases, and if they have the same frequency. If they are not in the same phase but have the same frequency, let ϕ be the angular phase difference. Representing the two maximum values by two lines and the phase difference ϕ by the angle between these two lines, we may find the maximum value of the sum by finding the vector sum of the lines representing the EMF's. The truth of this proposition can be seen from Fig. 240. Represent the instantaneous EMF's by e' and e'', and their sum by e; then $e = e' + e''$. E' and E'' represent the maximum values of e' and e'' respectively, and ϕ their phase difference. $e' = E' \sin \omega t$ and $e'' = E'' \sin (\omega t - \phi)$. From the figure it is evident that E is the vector sum of E' and E'', and it is also evident that E is the maximum value of e or $e' + e''$. Hence it is apparent that, while the instantaneous values of the EMF's in the circuit are added algebraically, we may find the maximum of the sum by adding the two component maxima as vectors, provided we know their phase difference.

392. Current in a Circuit Having Inductance and Resistance. —When a circuit has inductance L, we have seen that there is induced in it an EMF equal to L times the rate of change of current in the circuit. This EMF must be taken into account when we try to find the current in such a circuit. We can no longer write $E = IR$, meaning by E the applied EMF.

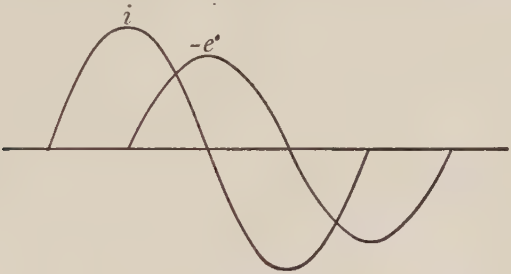


FIG. 241

This induced EMF may be found as follows, if the current is a harmonic function of the time. During one cycle the current rises to a maximum value, I, falls to zero, goes to a negative maximum $-I$, and rises to zero; hence the change in current is $4\ nI$ amperes per second. The average induced EMF is L times this average rate of change of current, *i.e.*, $4\ nIL$. Since $\omega = 2\pi n$ and $n = \omega/2\pi$,

$$\text{Average induced EMF} = \frac{4LI\omega}{2\pi}.$$

We have proved that when the EMF is harmonic, its average value is equal to $2/\pi$ times the maximum value. It is easily shown that the induced EMF is harmonic if the current is harmonic; hence we may write,

$$\text{Average induced EMF} = \frac{4LI\omega}{2\pi} = \frac{2E}{\pi},$$

from which Max. induced $E = \omega LI$. Since the instantaneous induced EMF is proportional to the rate of change of current, it will have its maximum value when the current is changing most rapidly; and this occurs when the current is passing through its zero value. Furthermore, by Lenz's law, it is negative when the current is increasing in the positive direction. If i is the current curve in Fig. 241, the curve marked $-e'$ is the curve of induced EMF. The maximum ordinate of this curve is ωLI. To produce the current i, we require two EMF's, one to counteract $-e'$, and the other equal to Ri and in phase with the current. These two EMF's are shown

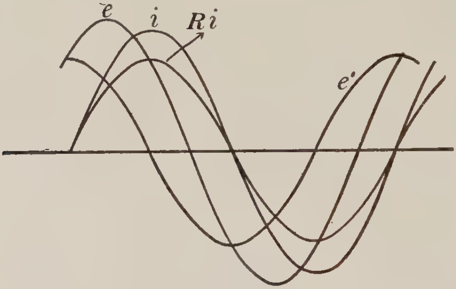


FIG. 242

in Fig. 242. The one which is equal and opposite to the induced EMF is represented by the curve marked e'. If the ordinates of this curve are added to the corresponding ordinates of the Ri curve, we obtain the curve e, which represents the EMF necessary to produce the current i. Since time is plotted to the right, it is evident that the current reaches its maximum value after the EMF has reached its maximum; in other words, the current lags behind the EMF in phase. The magnitude of this phase difference will be determined in the next paragraph.

393. Graphic Representation.—Since the two EMF's that were added to obtain E differ in phase by 90°, we may find E by the method discussed in the topic on the Circle Diagram.

E equals the sum of RI and ωLI, added as if they were vectors at right angles, *i.e.*,

$$E^2 = I^2\omega^2L^2 + I^2R^2,$$

and

$$I = \frac{E}{\sqrt{R^2 + \omega^2L^2}}.$$

The quantity $\sqrt{R^2 + \omega^2L^2}$ is called the impedance of the circuit and is measured in ohms. ϕ is the phase difference between E and I, and from Fig. 243 it is apparent that

$$\tan \phi = \frac{\omega L}{R}.$$

In a circuit having large inductance and small resistance, as in the case of the primary of a transformer without load, the resistance may be neglected and $I = \frac{E}{\omega L}$ approximately. The current then varies inversely as the inductance for a given frequency, and inversely as the frequency for a given inductance. When the frequency of the applied EMF is high, an enormous voltage is necessary to produce a small current in a circuit of moderate inductance. Under these circumstances it is also evident that $\tan \phi$ becomes very large and the angle of lag is nearly 90°.

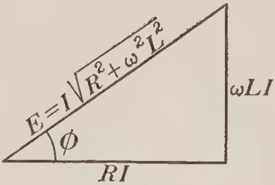


FIG. 243

394. The Oscillograph.—The oscillograph is an instrument that is used in studying phase relations and wave forms of alternating currents. It is like a D'Arsonval galvanometer, except that the moving coil is a single loop of wire stretched so tightly that its natural period of oscillation is very small, .001 of a second or less. The wire is fine and the mirror attached to it is very small; hence it has little inertia and can follow the rapid variations in the potential difference that is applied to it. The vibrating spot of light is thrown onto a moving piece of photographic film, this motion being at right angles to the direction of vibration. In this way, if the vibration is simple harmonic, a sine curve is traced. In any case the wave form is made visible. Two or more vibrating systems are used at the same time so that the phase relation of the current and EMF may be studied.

Fig. 244 is a photographic record, or oscillogram, for a non-inductive circuit, showing the EMF and current in the same phase, *i.e.*, $\phi = 0$, since $\omega L/R = 0$.

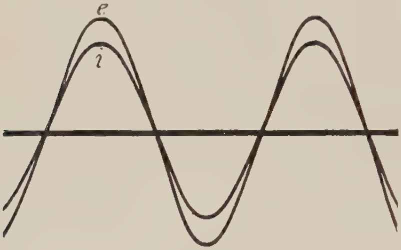


Fig. 244

Fig. 245 is an oscillogram showing the phase relation of current and EMF in a circuit having a very large inductance and a very small resistance. Tan $\phi = \omega L/R$ is large and ϕ is nearly 90°.

395. Power.—The average power expended in an inductive circuit is I'^2R, and is measured in watts, if I'^2 represents the R.M.S. value of the current in amperes, and R is the resistance in ohms. In order to express the power in terms of E' and I', we observe from the triangle of Fig. 243 that $RI' = E' \cos \phi$; hence

$$\text{Power} = I^2R = E'I' \cos \phi.$$

Cos ϕ is called the power factor and is usually expressed as a percentage. When L is large, or when ω is large, or when R is very small, ϕ approaches 90° and cos ϕ approaches zero; then the power expended upon the circuit also approaches zero. The term wattless current is used to represent the component of current which differs in phase from the EMF by 90°, since the net power used in producing this current is zero.

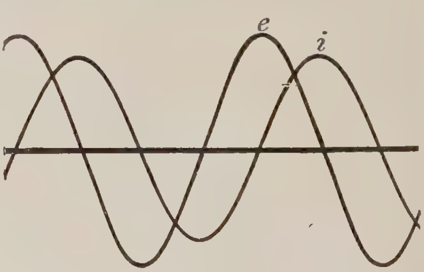


Fig. 245

We have referred to the fact that the primary winding of a transformer, being highly inductive, takes but little current when there is no load on the secondary. Now we see that even that current which does flow in the primary represents but small loss of energy because of the large angle of lag between it and the applied EMF.

396. Circuit Containing a Condenser.—A source of alternating EMF is applied directly to a condenser, as in Fig. 246, the resistance of the connecting wires being negligible. The condenser at every instant will contain a charge q sufficient to charge the plates to a potential difference equal to the applied EMF, e. By the definition of capacitance, $q = Ce$. When e is increasing most rapidly, q is also increasing at its maximum rate and the current in the connecting wires is a maximum. This occurs when e is passing through its zero value. From this it is apparent that the current is 90° ahead of the EMF in phase. During a quarter of a cycle the charge in the condenser rises from zero to its maximum value Q, and in the next quarter cycle it falls to zero. Thus a charge Q flows through the connecting wires four times per cycle, or $4n$ times per second; and hence the average current is $4nQ$ amperes. Substituting CE for Q,

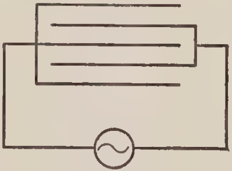


FIG. 246

$$\text{Average current} = 4nCE.$$

If e is harmonic, it may be shown that i will be also; hence

$$\text{Average } i = \frac{2I}{\pi} = 4nCE, \text{ from which } I = 2\pi nCE = \omega CE.$$

Multiplying both members of the equation by .707 gives the effective or R.M.S. value of the current in terms of the effective value of the EMF, *i.e.*,

$$I' = \omega CE', \quad \text{or} \quad I' = \frac{E'}{1/\omega C}.$$

$1/\omega C$ is called the reactance of the circuit and is measured in ohms. It will be observed that the current flowing to the condenser is directly proportional to the frequency, whereas in the highly inductive circuit it is inversely proportional to the frequency.

397. Condenser Circuit Having Resistance.—In this case the applied EMF must have two components, one whose maximum value is equal to RI to supply the drop in the resistance, while the other, of maximum value $\frac{I}{\omega C}$, differing in phase from RI by 90°, is the difference of potential between the plates of the condenser. The applied EMF is found by adding these two components as if they were vectors at right angles.

$$E = I\sqrt{R^2 + \frac{1}{\omega^2 C^2}}, \qquad \text{or} \qquad I = \frac{E}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}}.$$

The denominator of the last expression is called the impedance of the circuit. The diagram showing these relations is that of Fig. 247. The current is shown as leading the EMF by an angle ϕ.

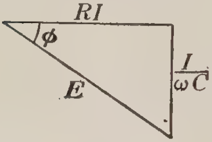


FIG. 247

398. Circuit Having Resistance, Inductance, and Capacitance in Series.—Fig. 248 shows the vector diagram for this case. $I/\omega C$ is 90° behind the current in phase, ωLI is 90° ahead of the current in phase, and RI is in phase with the current. These three components have a resultant marked E in the figure. ωLI is represented as larger than $I/\omega C$; hence the current lags behind the EMF by an angle ϕ. If ωLI were smaller than $I/\omega C$ the current would lead the EMF. The algebraic expression for the current in this case is evidently

$$I = \frac{E}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

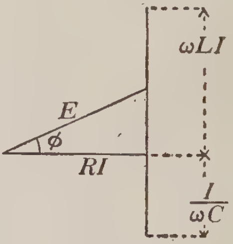


FIG. 248

If $\omega L = \frac{1}{\omega C}$ the last equation reduces to the form $I = E/R$, and the circuit behaves as if it had neither inductance nor capacitance. When this condition is fulfilled, the phase angle ϕ

becomes zero, and the current has the largest value possible for the given values of E and R.

PROBLEMS

1. The ordinary telephone condenser has a capacitance of two microfarads. Suppose twelve of these are connected in parallel to a 110-volt electric lighting circuit. The potential difference between the terminals rises from zero to 155 volts, falls to zero, then reaches 155 volts in the reverse direction, and then returns to zero. The condenser is alternately charged and discharged. The voltage goes through this complete cycle 60 times per second. Without reference to sign, find the average numerical value of the current in the connecting wires.

2. Draw a figure of a uniform field of induction B. Draw a side view of a thin square frame revolving in this field about an axis in the plane of the coil and at right angles to the field. Write an expression for the instantaneous value of the EMF in the frame, if it makes n revolutions per second. What is the maximum value of this expression? Next find the average EMF from the total flux cut per second. Show that the ratio of the average to the maximum EMF is $2/\pi$.

3. (a) Find the current used if 55 kw. are to be transmitted at 110 volts. (b) Find the current used if 55 kw. are to be transmitted at 2,200 volts. (c) If the same conductors were used in (a) and (b), compare the potential drops in the lines in the two cases. (d) Compare the percentage drops in the two cases, *i.e.*, the drop as a percentage of the applied voltage. (e) For a given size of wire and the same percentage drop, compare the distances to which the same power may be transmitted in the two cases.

4. A coil of wire having a resistance of 10 ohms and an inductance of .04 henry is connected to a 110-volt, 60-cycle, A.C. lighting circuit. (a) Find the current produced in the coil. (b) Find the phase difference between the EMF and the current. (c) Find the power used on the coil.

5. A condenser is put in series with the coil of the preceding problem. What is its capacitance if the phase difference between current and EMF is reduced to zero?

6. Find the impedance of the coil of problem 4 if the frequency is increased to 6,000 cycles per second. Estimate the error that would be made in the impedance if the resistance were neglected. How does this error compare with the corresponding error in problem 5?

CHAPTER XXVIII

OSCILLATING CIRCUITS

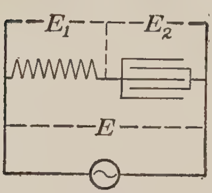


FIG. 249

399. Resonance.—As shown in the preceding paragraph, a circuit having inductance and capacitance in series will carry a maximum current for a given EMF if $\omega L = 1/\omega C$. The current is then limited only by resistance. Consider the circuit shown in Fig. 249. E represents the potential difference applied to the circuit, E_1 is the component potential difference between the terminals of the coil, and E_2 is the component applied to the condenser. If $\omega L = 1/\omega C$, the current is in phase with E, but it lags nearly 90° behind E_1 and it is 90° ahead of E_2. The vector diagram for these three potential differences is shown in Fig. 250. We have the phenomenon of a potential difference divided into two components, each of which is enormously greater than the whole. The condition assumed, viz., $\omega L = 1/\omega C$ is called the condition of resonance. The natural or free period is found by solving the equation for ω and then substituting $2\pi/T$ for it; thus:

$$\omega^2 = 1/LC, \text{ from which } T = 2\pi\sqrt{LC}.$$

By making either L or C variable, the circuit may be tuned to different frequencies.

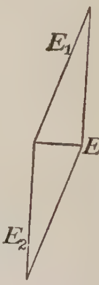


FIG. 250

400. Parallel Resonance.—Fig. 251 shows a condenser and coil of wire in parallel connected to a source of alternating EMF. The current in the coil is

$$I = \frac{E}{\sqrt{R^2 + \omega^2 L^2}};$$ and if R is negligible, $I = E/\omega L$.

This current lags behind the EMF by 90°. The current into the condenser is $I_c = \omega CE$, and it leads the EMF by 90°. Hence these two currents are opposite in phase; and if they are made equal by adjusting either L or C, we shall have currents flowing around the circuit consisting of the coil and condenser, which may be represented at some instant by the arrows in the figure. The current supplied from the source is the difference between the two currents; and as the two currents approach equality, the supplied current approaches zero. Thus we have an oscillatory current in the system, and the energy supplied to it is only sufficient to compensate for that which is lost in heat and in radiation. This is analogous to the vibration of the pendulum of a clock. Considerable energy is stored in the pendulum, but only enough is supplied at each oscillation to compensate for the loss due to friction. The condition of resonance here is the same as for the series arrangement since, when the two currents are made equal by tuning,

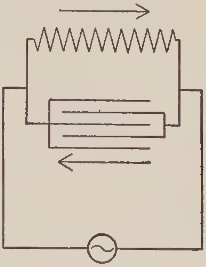


FIG. 251

$$E/\omega L = \omega CE, \quad \text{and} \quad T = 2\pi\sqrt{LC}.$$

When an oscillation is started and the system is left to itself without further supply of energy, the oscillation is said to be damped, its amplitude diminishes and the oscillation dies out. Plotting current as ordinate and time as abscissa, we may represent such a damped oscillation graphically. Fig. 252 is an oscillographic record of the oscillatory discharge of a condenser.

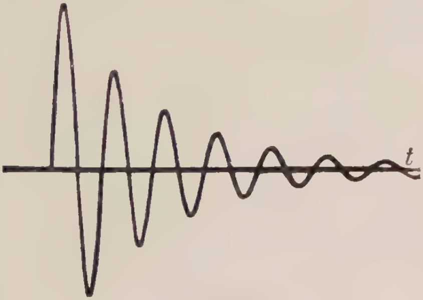


FIG. 252

401. High Frequency Generator.—Fig. 253 is a diagram of the apparatus that is usually employed to charge a condenser and produce an oscillatory discharge through a coil. The spark sets used in wireless telegraph systems differ from this only in having a transformer instead of an induction coil. A gap G is made between the coil and the condenser. An induction coil or other transformer supplies a charge to the condenser; and when the condenser plates are charged to a potential difference of several thousand volts, a spark occurs at the gap which virtually closes the circuit, and the condenser is discharged through the coil, *L*. The charge surges back and forth until its energy is all dissipated in heat or radiated into space. The frequency of the oscillation may be changed by adjusting either the capacitance of the condenser or the inductance of the coil. In this way the generating system may be tuned to resonance with another system, as, for example, an antenna or a long coil of wire. Such a coil is shown in the figure; it is coupled with the coil *L* in the oscillatory circuit so that currents are induced in it. The coil *L′* has both capacitance and inductance, and hence has a natural period of oscillation. It behaves like a closed organ pipe with the closed end at the top. The current flows into the coil, charging it to a high potential, and then flows out. During the next surge the charge is of the opposite sign. This is analogous to a condensation propagated into a closed organ pipe. At the end it is reflected, and then it is followed by a rarefaction. The fact that a node occurs at the top of the coil is apparent because sparks are seen to jump from it to a wire placed parallel to the coil and connected to earth. When the generator is tuned to a frequency about three times as high, the first harmonic is produced in the coil *L′*, and a second node appears at a point one-third of the coil length from the bottom.

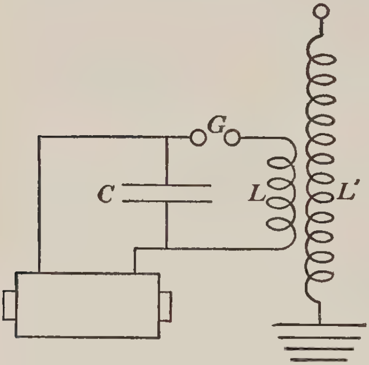


FIG. 253

The analogy between these electric oscillations and the vibrations in an organ pipe may be followed still further. When a condensation travels through an open pipe, upon reaching the end it goes on out into the air, while a rarefaction is at the same time reflected into the pipe. The node is at the middle of the pipe and there are antinodes at the two ends. To construct the analogous electrical apparatus, we have only to connect the upper end of coil L' to the wire leading to earth; then a charge flowing into the coil finds a way out when it reaches the upper end. The node at the middle of the coil is made visible by the glow discharge that takes place in this region. By doubling the frequency the first harmonic is produced, and the coil shows two nodes at about a quarter of the coil length from either end. Other higher harmonics may be obtained. A Geisler tube placed near a node will glow brightly, while if it is near an antinode it will remain dark. This furnishes a convenient means of exploring the coil to determine how it is vibrating. For demonstrations on the lecture table, the coil L' may be connected directly to the upper end of the coil L, instead of being electro-magnetically coupled to it as shown in the figure. In the oscillation transformer the coil L' surrounds the coil L and has many more turns than L has. When the primary and secondary are tuned to the same natural frequency, the secondary voltage may be very high. Several hundred thousand volts may be produced, and sparks a yard or more in length are thus obtainable.

402. Electromagnetic Radiation.—Consider a vertical antenna terminating at the top in a conducting sphere, and suppose a high frequency oscillation is set up in it. During the time that a current is flowing upward to charge the sphere, a magnetic field is established around the wire which may be represented by circular lines of force lying in a horizontal plane. At the same time, lines of electric force spread out from the wire, and their ends follow the charge as it moves toward the sphere. The lower end of each line of electric force is on the surface of the earth. These two fields move outward together through space with the speed of light (3×10^{10} cm per second). After the charge in the ball becomes a maximum, the current begins to flow down the wire and a new set of lines of magnetic force, reversed in direction, is started out behind the first. The upper ends of the electric lines follow the charge down the wire, so that finally both ends of the lines of electric force are on the earth. Fig. 254 represents both the electric and the magnetic fields established during the first quarter of a cycle, and Fig. 255 shows the phase relation of the two fields as they are propagated through space.

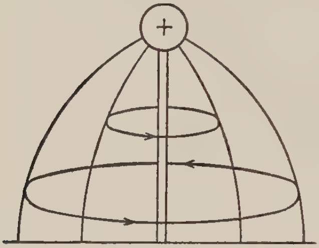


FIG. 254

403. Receiving Systems and Detectors.—In receiving and detecting an electromagnetic wave from a distant point, the principle of resonance is again applied. An antenna whose inductance or capacitance may be varied is tuned to unison with the radiating station. When resonance is attained, a very feeble radiation is sufficient to set up oscillations in the antenna. The spark telegraphic stations make use of apparatus similar to that described in § 401. A telegraphic key is put into the primary circuit of the transformer.

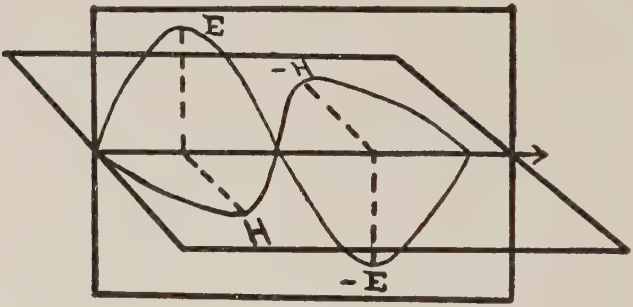


FIG. 255

The system radiates trains of damped waves which follow each other in such rapid succession that each train, if it could be heard, would constitute one of the vibrations in a musical note. The radiation is interrupted at the sending station by the operator at the key in such a way as to form the dots and dashes of the telegraphic code. Fig. 256 shows the simplest form of receiving apparatus. *A* is the aerial conductor. The tuning is done by sliding the contact point P up or down along the coil to vary the inductance. T is the telephone and C is a crystal detector. When a fine wire touches a crystal, it is often found that much more current will flow from the crystal to the wire than will pass through the contact in the reverse direction if the EMF is reversed. Some few crystals have this property in such marked degree that they are extensively used as rectifiers. When an alternating EMF is applied to them, they transmit the current in one direction but suppress the current in the other. It is for this reason that they are called rectifiers. When a train of electromagnetic waves reaches such a system as that shown in Fig. 256, oscillations will be set up in it, provided it is tuned to the frequency of the waves. An alternating potential difference will be

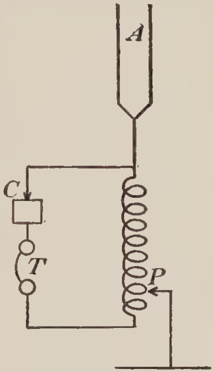


FIG. 256

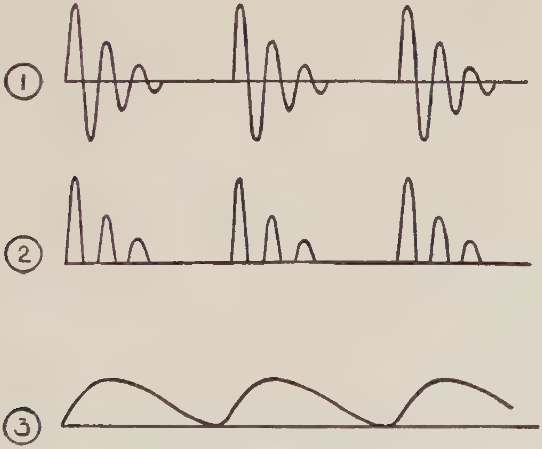


FIG. 257

set up between the terminals of the coil; and since the telephone is connected across the coil, this same potential difference will be applied to the telephone. However, no sound would be heard in the telephone since the frequency of the oscillation is far above the audible range. Even the fact that the waves come in trains does not alter this conclusion, since the impulse due to the wave train is zero, there being as much tendency for the telephone diaphragm to move one way as the other. If, however, a rectifier is put in the circuit with the telephone, one-half of the wave train is cut off, as shown in the second part of Fig. 257, and the high frequency impulses upon the diaphragm of the telephone are all in one direction. These blend together into a single impulse for each wave train, as shown in the third part of the figure. Since there are several hundred wave trains per second, a musical note may be heard in the telephone. When the key at the sending station is operated, the musical note becomes intermittent and thus the signals are made.

404. The Electron Tube.—Crystal rectifiers and electrolytic rectifiers have been used successfully as detectors, but the electron tube has many advantages over the simple rectifier and has opened up possibilities in radio work that were not suspected when the tube was first put into use. The tube, when used as a simple rectifier, has two electrodes, one a wire heated to incandescence by an electric current and known as the filament, the other a sheet of metal called the plate. The electrodes are sealed into a tube from which the air is afterward exhausted. Incandescent bodies emit electrons very much as a hot liquid throws off a vapor, and if the plate is positively charged the electrons are attracted to it; thus they become the carriers by means of which the current is conveyed across the evacuated space in the tube. This thermionic emission is very copious in the case of certain oxides. A wire coated with calcium and barium oxides need only be heated to a dull red heat to emit electrons in large numbers. Tungsten wire that is permeated with thorium is used in the construction of many of the recent tubes, and the cæsium-coated filament is being used at temperatures so low that the wire does not emit visible light. If the plate is connected to the negative terminal of a battery and the filament to the positive, the electrons are repelled by the plate and no current is conducted through the tube. Thus it appears that the tube is a rectifier and, in fact, a more nearly perfect rectifier than any other that has been discovered. The two-electrode tube is known as the kenotron. It could be substituted with good results for the crystal rectifier in radio reception, and probably would be if the three-electrode tube were not so much more effective as a radio detector. Fig. 258 illustrates the kenotron connected to a source of alternating EMF. During half the cycle current flows in the direction indicated by the arrow. During the other half of the cycle no current flows between the plate and the filament.

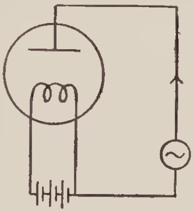


FIG. 258

405. The Three-Electrode Tube.—The next step in the development of the electron tube consisted in placing a grid or mesh of fine wire between the filament and the plate. Since the electrons must pass through the grid in order to convey current through the tube, it is evident that the potential of the grid may determine the current. If the grid is charged negatively so that its potential is considerably lower than that of the filament, it will repel the electrons and prevent their reaching the plate; while if it is raised to a potential higher than that of the filament it will accelerate their motion and thus increase the current. An experiment which strikingly illustrates this function of the grid may be explained by means of Fig. 259. The grid *G* of the tube is connected to a vertical wire two or three feet long, marked *A* in the figure. A battery and a relay are connected between the filament and the plate. The relay armature is drawn forward with considerable force as the filament is heated and the tube becomes a conductor. The relay is so arranged that the local circuit is open when the

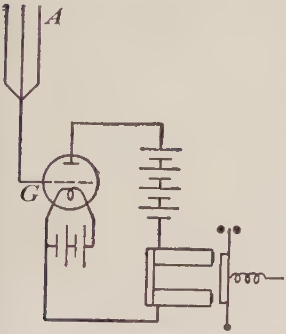


FIG. 259

armature is drawn forward by the magnet, and is closed when the armature is released. The local circuit is not shown in the figure. When a charged rubber rod is brought within two or three feet of the wire connected to the grid, the grid becomes negatively charged and the battery current is reduced to a small value or perhaps to zero. The relay armature is then released, thus closing the local circuit, and the current in the local circuit furnished by a local battery then operates whatever apparatus is connected in the circuit. A bell may be rung, lamps lighted, or a motor started. The energy supplied to the grid is too small to be measured; yet it is evident that practically any amount of power may be under the control of any agent that can vary the potential of the grid by a few volts. An electromagnetic wave from a distant source is capable of producing this variation of grid potential, and thus may be made the means of controlling large amounts of energy supplied by a local generator.

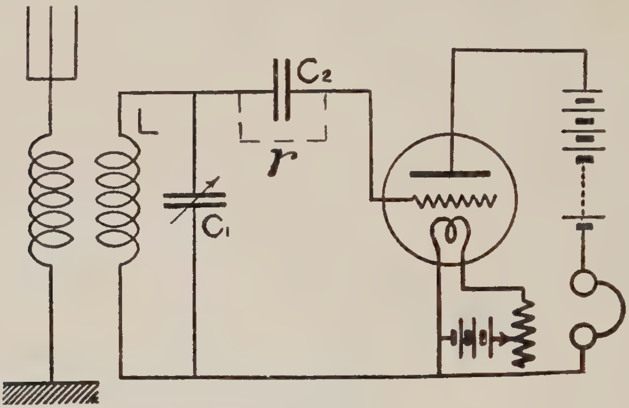


Fig. 260

406. The Tube as a Detector.—The essential features of a receiving system in which only one tube is used are shown in Fig. 260. A tuned antenna is coupled to the oscillatory circuit L, C_1. One side of the condenser C_1 is connected to the filament, and the other side through a grid condenser to the grid of the tube. When the system is set in oscillation, the grid potential will rise and fall with the oscillations, causing corresponding fluctuations in the current through the tube and telephone. However, this does not describe the detector action of the tube, since high frequency fluctuations in a telephone current produce no audible effect.

In order to understand the detector action, we must consider the effect of the grid condenser C_2, which is placed between the oscillatory circuit and the grid. A grid leak r, of two million ohms or more, is shunted around the grid condenser. The grid, being surrounded by an atmosphere of electrons, will take on a small negative charge. When an oscillation is started in the circuit LC_1 the grid fluctuates in potential. During that part of the time when its potential is higher than the normal value, it will take on more electrons, thus making its average potential more negative than the normal value. This decrease of the average grid potential diminishes the current through the tube and the telephone. When a wave train has faded out, the negative charge that has accumulated on the grid leaks off through the grid leak r, and the normal grid potential is restored before the next wave train arrives. Thus it is seen that each wave train causes a temporary decrease of telephone current, and hence a sound is produced in the telephone whose frequency is that of the wave trains. Fig. 261 illustrates these effects.

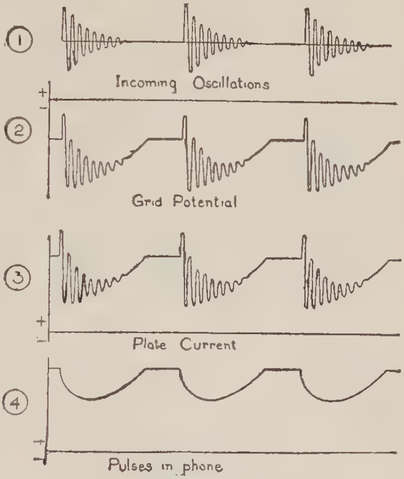


Fig. 261

The explanation of the use of the detector tube for the reception of continuous modulated waves such as are used in speech reproduction is very similar to that just given. The apparatus differs in no respect from that described.

407. Amplification.—If the primary winding of a step-up transformer is put in place of the telephone in Fig. 260, the secondary voltage of the transformer can be applied to the grid of a second tube. If the telephone were put in the plate circuit of this second tube, its current fluctuations would be considerably larger than when it was in the plate circuit of the first tube. A set of this kind is said to have a detector and one stage of audio-frequency amplification. Two or three stages are frequently used for amplifying weak signals.

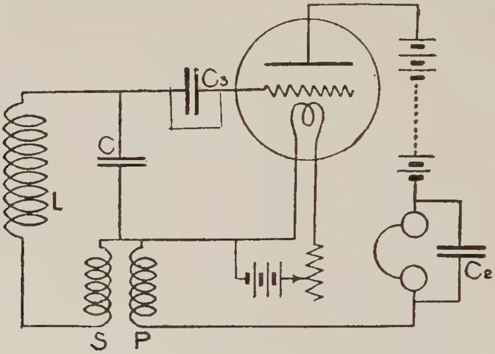


Fig. 262

408. Regenerative Amplification.—If the system of Fig. 260 is modified by inserting a coil in the plate circuit and so placing it that it acts inductively upon the coil in the oscillatory circuit, we have what is known as a regenerative system. The diagram is shown in Fig. 262. The battery current fluctuates in unison with the oscillations in the circuit LC. This fluctuating current in passing through the coil P acts inductively upon the coil S, thus adding to the amplitude of the oscillations in that coil. This increased amplitude results in a larger amplitude of oscillating grid potential which in turn increases the oscillatory component of the plate current. In this way the signal is strongly amplified and the detector action may be increased as much as fifty-fold.

409. The Tube as a Generator.—If the coils S and P in Fig. 262 are closely coupled so that energy is fed into the oscillatory circuit from the battery circuit as fast as it is dissipated, the oscillations become self-sustained and the system becomes a high frequency generator. There are many types of high frequency generators, but the same principle underlies them all. The tube generator is the essential feature of the modern broadcasting station. The generator forces a large antenna into oscillation, and a considerable part of the input energy is radiated as an electromagnetic wave. The high frequency wave is modulated by a telephone transmitter so that it carries with it fluctuations in intensity that correspond to the vibrations of the sound to be transmitted.

410. The Wave Meter.—Since a circuit made up of a condenser and a coil of wire in series is an oscillatory circuit, having a definite period, such a circuit will show the property of resonance, and there will be established in it electrical oscillations, provided it receives electromagnetic waves from some other oscillatory system having the same frequency. If the condenser is variable, such a circuit may be tuned to the same frequency as that of the circuit furnishing the waves. A thermal ammeter in the circuit will show a maximum current when the two systems are of the same frequency. The variable condenser may be furnished with a scale calibrated directly in wave lengths Such an instrument is called a wave meter. It is primarily a frequency meter, but the electromagnetic waves travel with the speed of light, 3×10^8 meters per second; and hence for every frequency, there corresponds a definite wave length. Since $\lambda = VT = V/n$,

$$\lambda = \frac{3 \times 10^8}{n} \text{ meters.}$$

One of the wave lengths prescribed by law for amateur radio operators is in the neighborhood of 160 meters. Evidently this corresponds to a frequency of 1,875,000 cycles per second.

411. Photo-Electric Effects.—If an electron falls through a potential difference of 16.4 volts it acquires sufficient energy so that, in collision with a hydrogen molecule, it may dislodge an electron from the atom. The minimum difference of potential required to produce this ionization is called the ionizing potential of the gas. It differs for different gases. In a similar way, if a solid substance, such as a metal plate in a vacuum tube, is bombarded by an electron stream, electrons are torn from the metallic atoms and are free to move to another electrode in the tube. These are called secondary electrons. The amount of secondary emission increases as the kinetic energy of the primary electrons is increased; and with 100 volts or more applied to the plate, the number of electrons given off from the plate may be several times as great as the number in the bombarding stream.

These phenomena, together with those connected with thermionic emission, would lead one to feel no surprise on finding that, under certain conditions, electrons are liberated from a metal surface if light falls upon it. Electrons set free in this way are called photo-electrons, and the tube in which photo-electric emission occurs is called a photo-electric cell. Before describing the cell, some facts that have had a profound influence in the development of modern physics should be mentioned.

A certain amount of energy, ω_o, is required to remove an electron from a metallic plate of given material. It is called the work function of the substance. If more than this amount of energy is absorbed by an atom, the excess becomes kinetic energy of the moving electron as it emerges from the plate, and is represented by $\frac{mv^2}{2}$. The fact that has been of such importance in modern physics is this: the energy absorbed by an atom from a beam of light is directly proportional to the frequency of the radiation and is in no way related to the measured intensity of the beam. This is stated in "Einstein's photo-electric equation" thus:

$$\frac{mv^2}{2} = h\nu - \omega_o,$$

i.e, the kinetic energy of the electron equals the total energy supplied to the atom minus the energy used in getting the electron out of the metal. ν is the frequency of the light wave and h is a fundamental constant of nature, known as "Planck's constant," because Planck was the pioneer who first expounded the "quantum theory." The product $h\nu$ is called the quantum of energy related to light of frequency ν. When the energy is expressed in ergs and the frequency in vibrations per second, the numerical value of h is 6.56×10^{-27}. From the equation it is evident that if the work function ω_0 is as large as $h\nu$, the velocity v becomes zero; *i.e.*, there is a definite frequency below which the energy supplied to any one atom is insufficient to remove an electron, and there is no photo-electric emission, even though the total energy supplied at the low frequency is very great. If, during the war, an army had no gun that would carry to the height of a given enemy airplane, nothing was gained by increasing the number of guns. So here, if the energy that the atom can absorb is limited to $h\nu$, there will be no emission unless $h\nu$ is as great as the energy needed to remove a single electron. On the other hand, if ν is such a frequency that emission occurs at all, an increase of the intensity of the light beam will increase the number of electrons emitted.

The phenomena connected with photo-electric emission have furnished remarkable evidence that radiation, such as light, is not transmitted by a continuous wave, as was formerly supposed, but that it travels as "light quanta," or very small portions of energy isolated from each other. A quantum for violet light of wave length .00004 cm consists of

$$6.56 \times 10^{-27} \times .75 \times 10^{15} = 5 \times 10^{-12} \text{ erg.}$$

For most metals the energy required to remove an electron from the atom is so great that only very high frequency radiations are able to cause the separation; but in the case of the alkali metals, potassium and sodium, the work function is so small that radiations having the frequencies of visible light are effective in producing photo-electric emission. For this reason the photo-electric cells that are most widely used are made with one or the other of these metals.

412. The Photo-Electric Cell.—The applications of photoelectric phenomena in apparatus and instruments have excited popular interest to almost as great an extent as the scientific aspects have interested physicists. It is the photo-electric cell that has made possible television, the transmission of pictures by wire, and the talking moving picture. A diagram of the photo-electric cell is shown in Fig. 263. It is made of glass or quartz, with a wire loop near the center for collecting the electrons. In making the cell, sodium or potassium is put into the bulb. It is then connected to a pump and the air is removed. Next the metal is evaporated by heat and then allowed to condense on the glass surface so as to cover a large fraction of its area. A little hydrogen is then admitted and an electric discharge is sent through the tube from one electrode to the other. This causes the formation of sodium hydride, assuming sodium to be the metal used. The light sensitivity of the hydride is much greater than that of the pure metal. The hydrogen is then pumped out and argon admitted, after which the tube is sealed and removed from the pump. When the electrons are emitted by the action of the light they move through the tube as a result of the high potential difference applied between its electrodes. They collide with argon atoms and produce other electrons by ionization; thus the current is much larger than it would be if only the photo-electrons were acting as carriers of the charge.

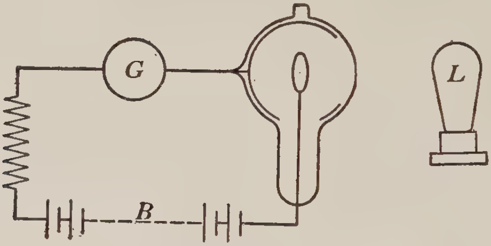


FIG. 263

In the figure, a circuit is shown by which the function of the cell may be illustrated. *L* represents an electric lamp which illuminates the light-sensitive electrode, and *G* is a galvanometer for showing the variations of current. The EMF of the battery should be between 100 and 200 volts. The current is practically zero when no light reaches the interior of the cell, and it increases proportionally as the intensity of illumination increases, provided the wave length is kept the same. If an opaque object—a pencil, for example—is brought between the lamp and the cell, the galvanometer shows a decreased deflection due to the shadow cast upon the sensitive surface. As in other electronic phenomena, the effect is instantaneous; there is a change in current corresponding to every change in the intensity of illumination. It is easy to see how such a cell could be used in a great variety of ways.

REVIEW PROBLEMS

1. The velocity of an electromagnetic wave is 3×10^8 meters per second. Find the frequency of a wave whose length is 360 meters.

2. A variable condenser having a range from 10 to 500 micromicrofarads is connected in series with a coil having an inductance of .5 millihenry. Over what range of radio wave lengths may the circuit be tuned?

3. The potential difference between the terminals of a generator is 220 volts. The power delivered to the lines is 50 kilowatts. What resistance should the wires have in order that the potential difference at the load may be 200 volts?

4. A 10 h.p. motor requires 220 volts. The generator that operates it produces 230 volts. What is the maximum resistance that the connecting wires may have, if the voltage at the motor is to be kept at 220 volts or more?

5. Fifty incandescent lamps in parallel constitute a group at a distance of 200 ft. from a center of distribution at which the voltage is 110 volts. Each lamp takes $\frac{1}{2}$ an ampere. Find the maximum allowable resistance of the conductors, if 2 per cent drop is specified as the maximum line drop.

6. The resistivity of aluminum is 1.82 times as great as that of copper. The specific gravity of aluminum is 2.6, and that of copper is 8.9 When copper is worth 30 cents a pound, what must be the price of aluminum to compete with copper as an electrical conductor?

7. The combined resistance of trolley line and rail return of a car line is .4 ohm per mile. What is the drop in potential in the line if a car 2 miles away is using 50 amperes? What is the power loss due to this drop in potential?

8. A 16 C.P. 110-volt carbon lamp takes a current of .45 ampere. Find its resistance while hot. A 16 C.P. 220-volt lamp would have what resistance and would take what current if it were equally efficient?

9. If a current of 10 amperes flows through a divided circuit consisting of an 8-ohm coil and a 12-ohm coil in parallel, what current flows in each wire, and what is the potential difference between the extremities of the coils?

10. Six cells of storage battery, having a resistance of .6 ohm and an EMF of 12 volts, are to be charged with a current of 8 amperes from a 112-volt lighting circuit. What resistance should be put in circuit with the battery to keep the current at the desired value? How efficient is the charging process? Answer the same questions if the battery consists of 40 cells in series, and has a resistance of 4 ohms.

11. Draw the following figure: *ABCD* is a long straight wire having a resistance of .2 ohm per foot and carrying a current of $\frac{1}{2}$ an ampere. A Daniell cell of EMF 1.1 volts has one terminal connected to *B* and the other terminal to a galvanometer. The other terminal of the galvanometer is connected to *C*. This point of contact, *C*, is made to slide along the wire until the galvanometer shows no deflection. What is now the distance between *B* and *C*?

12. A cell having an EMF of 1.1 volts and an internal resistance of 1 ohm, is connected in parallel with a cell having an EMF of 1.5 volts and internal resistance of 2 ohms. The external circuit is a wire of 3 ohms resistance. Determine the current in the wire and the part of it contributed by each cell. (Suggestion: Form three simultaneous equations in accordance with Kirchhoff's laws.)

LIGHT

CHAPTER XXIX

RECTILINEAR PROPAGATION

413. Nature of Light.—A lighted room discloses one or more sources of light, and objects, such as the walls, chairs, etc., become evident through the light received from these sources. Similarly, during the day things can be seen because they have received light directly or indirectly from the sun. In addition to rendering things visible, the source of the light often warms them. Energy in some way passes from the source to the object, whether lighted or warmed.

Newton's theory was that the sources of light emitted corpuscles of minute size which acted as the carriers of the energy. Huygens, on the other hand, believed that this energy was transmitted by means of wave motion, in a medium which he imagined permeated all space, and which has been known as the ether.

Many of the phenomena of light are readily explained by either hypothesis. It appeared, however, that in accounting for the phenomena accompanying the passage of light from one medium into another, as from air to water, the corpuscular theory required that light travel faster in water than in air, while the wave theory demanded the opposite. When experiment conclusively demonstrated that the speed in water was less than in air, the wave theory was accepted, and it has been uniformly successful in accounting for the phenomena connected with the propagation of radiant energy.

Only in recent years has material progress been made in analyzing the phenomena connected with the acts of emission and absorption of light and other forms of radiant energy. Evidence shows that the emitters of light are atoms or molecules, that is, the energy of atoms or molecules alone is transformed into light energy. Thus it is not the filament of a lamp as a whole which gives out the light, but rather the individual atoms which make up the filament. Similarly, when light strikes the surface of objects and some of its energy is transformed into heat, as when the sunlight warms things, it is only the atoms and molecules composing the bodies which act as agents for changing this energy into heat energy.

These acts of atomic and molecular emission and absorption in their many different aspects are not readily explained by the wave theory of light; rather they are more easily understood in terms of some kind of a corpuscular theory. Since some of the phenomena of the propagation of light cannot yet be explained by any corpuscular theory, there is created a situation in which the propagation of light with its attending phenomena is accounted for by the wave theory, and the acts of emission and absorption, together with similar phenomena, by a corpuscular theory. The reconciliation of these opposing aspects of the theory of light is one of the major problems of present science.

414. Beams and Pencils.—When a point source of light L is placed behind a screen S, the cone of light which passes through any aperture in the screen has a form which immediately suggests that light is propagated in straight lines.

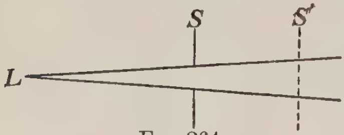


FIG. 264

The cross-section of the cone by any normal plane, as S' in Fig. 264, has the same form as the opening in S, whether this be circular, square, or any irregular shape. Its area is greater than the area of the aperture in the ratio of the square of the distances of S' and S from L. This cone, as it proceeds from L through the screens, is called a *beam* or *pencil* and, in this case, a *diverging* pencil. As will be seen later, it is possible, by using the proper optical devices, to cause a beam of the same form as the one above to proceed in the opposite direction. In this case the beam is a *converging beam*.

415. Shadows.—The familiar shadow has its explanation in the rectilinear propagation of light. This is also the idea back of the aiming of guns by means of sights. It is also the reason for turning one's eyes toward the object to be looked at. So firmly is this property of the propagation of light established that the prediction of Einstein that light traverses a curved path when passing near the sun has attracted universal interest and has resulted in the most elaborate experimental tests to determine its truth.

This fundamental property of light radiation is most obviously explained upon the basis of a corpuscular theory. It can be explained very completely, however, by means of the wave theory, as will appear later.

416. Images by Small Apertures.—When the source of light is an object of some size instead of a point, then each point of the object should be regarded as the source of a diverging beam passing through the aperture. If the object in front of the screen S in Fig. 265 contains a point g, which is emitting green light, then a green beam passes through the aperture in S and makes a green spot at g' on a white screen S'. Similarly, a red point r in the object would send out a red cone which would make a red spot on the screen S' at r'. Likewise for a point emitting blue light.

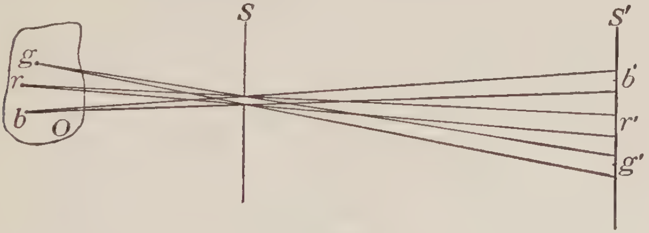


Fig. 265

It is evident that every point in the object O will be represented on the screen S' by a spot of light, and that this spot of light will have the same color as the emitting point in the object. If the point is emitting brilliantly the spot is intense; if the point in the object is dull, the spot on S' is dull. The spots on

S' are arranged in the same relative positions as the emitting points in O, except that upper points in O are below on the screen, and points in the right-hand side of the object are found on the left-hand side of the screen.

If the aperture in S is so small that the spots on S' do not overlap too much, there appears on S' an *image* of the object O. This image reproduces its object with respect to the positions of its parts, their color, and the relative brilliancy of their illumination. The size of the image is to the size of the object as their relative distances from the screen S. Thus, if the aperture is very small, the line gb in the object produces the line $b'g'$ in the image, and their lengths are proportional to their distances from S.

It will be seen that if the aperture is smaller the overlapping of the spots on S' is less and the image is more distinct. The larger the aperture, the brighter the image. Thus it is not possible for an image to be formed in this way which is both bright and distinct.

417. Law of Intensities.—As a sound loses intensity as the distance from the source increases, so also in the case of light, the light is fainter with increasing distance from the source. The intensity of light at a point is measured by the light energy passing per second through a unit area at that point normal to the direction of the beam. It will be recalled that the intensity of a sound at any point is measured in the same way.

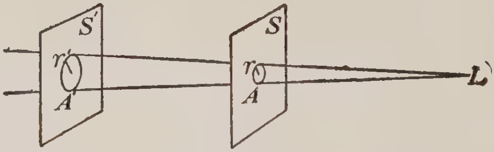


Fig. 266

Reference to Fig. 266 will show that the intensity of light from a point source varies inversely as the square of the distance from the source. This law is commonly spoken of as the *Inverse Square Law.* Thus, the energy passing through the circular aperture A in screen S per second, also passes through the circular aperture A' in screen S' per second. If S' is twice as far from L as S is, then the geometry of the figure indicates that the radius r' is twice the radius r, and the area of the aperture A' is four times that of A. The energy per unit area in A' is therefore 1/4 that in A, or, doubling the distance, the intensity is reduced to 1/4, an inverse square relation.

In general, if I is the intensity at a distance L from the source, and I' at a distance L', then $\frac{I}{I'}$ as $\frac{L'^2}{L^2}$. This law assumes that the energy of the beam is not altered in amount by the medium through which it is passing.

When the source of light is not a point, this relation does not hold, particularly if the source is large and the intensity near it is measured. Thus it would not be found that the intensity of the light from a burning house would decrease to 1/4 by increasing the distance from 50 to 100 ft. With the sources of light used for general illumination, the law holds quite rigorously at comparatively short distances from the lights.

418. Photometry.—The enormous increase in artificial lighting has made it necessary to be able to compare accurately the various types of lamps in use. While it is readily possible to tell by the eye when one lamp gives more illumination than another at a given surface, it is quite impossible to judge the degree of difference by this simple method. When adjacent surfaces are illuminated, each by a separate lamp, it is possible to judge equality of illumination by the eye with great accuracy when the colors of the lights are the same. This characteristic of the eye, together with the *Law of Inverse Squares*, is used to make such comparisons in an instrument called a Photometer.

The definition of intensity would indicate that a light of unit intensity should be one that could deliver light energy at the rate of an erg per second to each square centimeter of surface 1 cm distant from the source. The standard source is, however, a lamp which can be made to reproduce a light of constant intensity. Its brilliancy is that of the average candle, and the lamp is called a *one candle power* source. The illumination which it gives at a distance of 1 ft. is unit illumination, and is called a foot candle. The intensities of all other lights are measured in terms of this lamp.

A simple type of photometer is shown in Fig. 267. Two pieces of paraffin, *A* and *B*, are separated by a sheet of tin foil *TF*. With the standard lamp illuminating the piece *A*, and the lamp to be calibrated illuminating the piece *B*, the eye has only to decide when the two pieces appear equally illuminated. This equality of illumination is obtained by properly adjusting the distances of the standard lamp and tested lamp from the surface *TF*. Thus, if the standard lamp is at a distance S_1 ft.; the illumination at *TF* is $\frac{1}{S_1^2}$ of what it is at unit distance. Since this is by definition unity, the illumination of the standard lamp at *TF* is $\frac{1}{S_1^2}$. That of the other lamp, whose distance is S_2, ft., is $\frac{x}{S_2^2}$, where x is the illumination at unit distance from the lamp of unknown brightness, and is therefore the quantity to be measured. Since both lamps give the same illumination at *TF* when all adjustments are made,

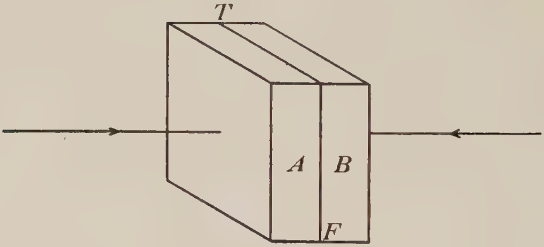


FIG. 267

$$\frac{1}{S_1^2} = \frac{x}{S_2^2}, \text{ or } x = \frac{S_2^2}{S_1^2} \text{ candle power.}$$

While x is the illumination at unit distance from the tested lamp, it may properly be called the candle power of the lamp, since x standard candles at the same distance will produce the same illumination as the lamp.

When the two sources are of different colors it becomes quite impossible to compare their brightness by this simple method, and other means have to be employed.

PROBLEMS

1. Through a small aperture in a window, a building 200 feet away produces an image 4 inches high on a screen 3 feet from the window. How high is the building?

2. The sun is 93 million miles away and its diameter is 865,000 miles. What will be the size of the sun's image on a screen 10 feet from a small aperture, the proper alignment having been made?

3. A tree casts a shadow 200 feet long. How can a boy remaining on the ground measure the height of the tree with a yard stick?

4. What is the candle power of a lamp when it must be at a distance of 2.5 meters from the photometer shown in Fig. 267 in order to give the side toward it the same illumination as the standard lamp at 1 meter gives its side?

5. Where must the photometer be placed between two lamps of 400 and 600 candle power respectively to show its sides equally illuminated, the lamps being 5 meters apart?

6. Two lamps of 100 and 60 candle power respectively are placed 50 feet apart. At what point or points on a line with the lamps will a screen be equally illuminated by each lamp?

7. Three candles are placed near together on a table and 200 cm from this group is another group of two candles. A screen is set up between the two groups in such a position that when all the candles are lighted the two sides of the screen are equally illuminated. How far from either group is the screen?

CHAPTER XXX

REFLECTION

419. Types of Reflection.—When light is incident upon a surface separating two media, some of the radiant energy is thrown back into the first medium, or reflected. The remainder enters the second medium. Here a portion or all of it is absorbed, that is, transformed into heat. If the second medium is transparent, a fraction may pass through.

Of the reflected portion, two kinds are recognized. When the surface is rough or not optically smooth the reflection is irregular. Thus, an incident beam having a definite direction becomes scattered in all directions by irregular reflection. Objects which are not self-luminous become visible by irregularly reflected light. Regular reflection occurs at surfaces which are sufficiently smooth so that a definite beam retains a definite configuration. A mirror is a familiar example of such a surface. Sunlight is regularly reflected from quiet water.

420. Regular Reflection.—When a ray of light approaches a surface, the angle between the ray and the normal to the surface is called the angle of incidence. If the ray is reflected, the angle between the reflected ray and the normal is called the angle of reflection.

The *Law of Reflection* states that the angle of reflection is equal to the angle of incidence, and that the two angles lie in the same plane. The law is equally true whether the reflecting surface is plane or curved.

To an observer looking into the mirror, Fig. 268, L' appears to be the only source of light, and all beams which leave the mirror seem to have started from L'. Thus the beam $OO'SS'$ is the real portion of the apparent beam $L'SS'$. LOO' must be the first part of the actual beam. These results of observation find an explanation in either the corpuscular theory or the wave theory. The first requires only a perfectly elastic rebound when the individual corpuscles composing the beam *LOO′* strike the mirror surface. The wave theory accounts for the phenomena equally well if Huygens' Principle is applied here as it was in sound. (See §179.)

Applying the wave theory, the arcs with *L* as center, lying between *L* and *OO′*, represent the incident wave fronts, and the arcs with *L′* as center represent the reflected waves leaving the mirror. The line of propagation of energy in light, as in sound, is normal to the advancing wave fronts. Accordingly any line starting from *L* and cutting all the arcs normally is a possible line of flow of light energy. Such a line is usually called a ray; thus *LOS* and *LO′S′* are two rays which serve to bound the beam *LOO′SS′*. Such a beam contains an infinite number of rays. *L′*, which appears to be the source of the light reflected from the mirror, is an image of *L*, the real source. It is a virtual image since no light actually passes through it.

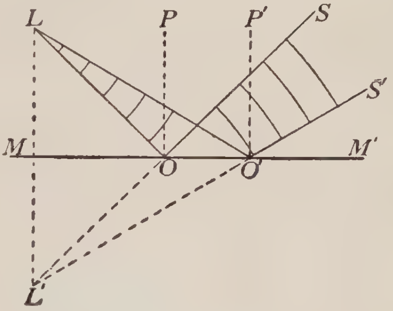


Fig. 268

These ideas developed with the aid of a plane mirror could have been developed with curved mirrors, but not so simply.

421. Image of a Point Produced by a Plane Mirror.—If *AB* in Fig. 269 represents a plane mirror, and *P* a luminous point in front of the mirror, a virtual image of *P* is formed behind the mirror. The law of reflection enables us to find the position of this image, *Q*. Let *PM* be a ray that is reflected to *S*, and *PN* another ray reflected to *T*. These rays are in the boundary of a cone of rays that would seem to an observer to come from *Q*. Thus *Q* is the image of *P*. The angles in the figure are represented by the small letters.

$$a = b \text{ and } g = h, \text{ by the law of reflection.}$$
$$a + b + c = d; \text{ why?}$$
$$l = k; \text{ why?}$$

The triangle PMN = triangle QMN, since they have a side and two adjacent angles of one equal to the corresponding parts of the other. Then $PM = QM$ and $PN = QN$.

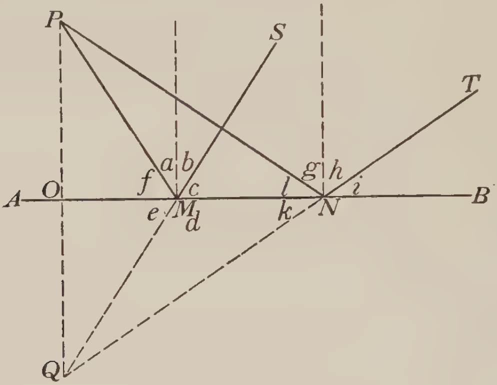


FIG. 269

Since two points each equally distant from the extremities of a straight line determine the perpendicular bisector of the line, it follows that $PO = QO$ and AB is perpendicular to PQ. Thus it is seen that the image of a point produced by a plane mirror is as far behind the mirror as the point is in front of it, and the two points are on a line perpendicular to the mirror.

422. Path of Reflected Ray.—An object AB, in Fig. 270, is placed in front of a plane mirror MN. The image of each point of the object is as far back of the mirror as the point is in front of it and is on a line through the point perpendicular to the mirror. Thus $A'B'$ is the image of AB. If the image is viewed from E, the light rays by which the image is seen lie in a cone bounded by $A'E$ and $B'E$. These boundary rays actually come to the eye from points H and K on the mirror, and they reach the mirror directly from the points A and B. Hence the paths of the rays are AHE and BKE respectively. Other rays of the cone may be found in the same way. If the eye were placed in a different position, totally different rays would be found to convey the impression to the eye.

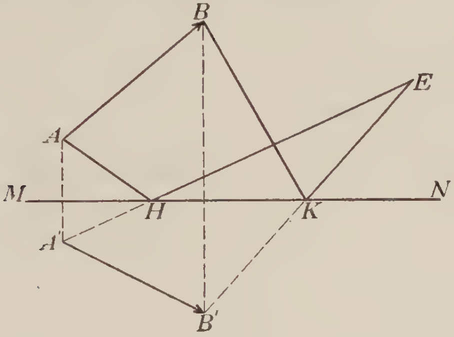


Fig. 270

423. Reflection at a Spherical Surface: Caustics.—In light, as in sound, when a plane wave is used, the lines of energy flow, or the rays, are all parallel to each other and normal to the wave fronts. Thus, in Fig. 271, plane waves whose fronts, not shown, are normal to *PP* are incident upon a spherical mirror whose center of curvature is *C*. It is convenient to treat the reflection by means of the rays which are parallel to the mirror axis, *PP*. The ray *AB* makes an angle of incidence with the normal, *CB*, at the point of incidence equal to *ABC*. The angle of reflection, *CBD*, is constructed equal to the angle of incidence. It will be noted that the successive incident rays, as the axis *PP* is approached, have smaller angles of incidence and reflection. This results in the reflected rays cutting the axis *PP* successively further from the vertex of the mirror *P*, until a point midway to *C* is reached. This point, to which all rays parallel to *PP* incident upon the mirror near the point *P* are quite accurately reflected, is called

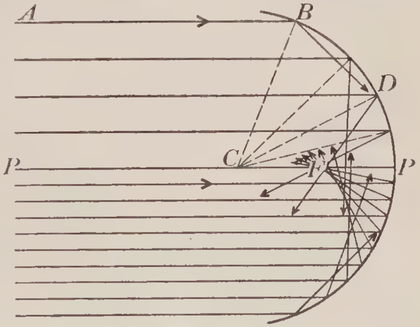


Fig. 271

the *principal focus* of the mirror. As already mentioned, this focus is half way between the center, C, of the mirror, and the vertex of the mirror, P.

While the parallel rays incident near the vertex are brought to a fairly definite focus at the principal focus, the rays farther out cut one another after reflection along a curved path, shown more clearly in the lower half of the figure where more rays are used. The intersections of the successive reflected rays form a series of foci, which would form a continuous curve were all the rays to be employed. This curve, with its duplicate above the axis, is called a *caustic curve.*

It will be evident that if a sharp focus is to be obtained with a spherical mirror, only a small portion of the surface near the vertex *P* should be used. If the outer reflected rays are also to be brought to focus at the principal focus, it will be necessary to change the shape of the mirror. A parabolic mirror has the proper shape to accomplish this purpose.

424. Position of Principal Focus.—The principal focus of a spherical mirror is the focus obtained when rays parallel to the principal axis of the mirror are incident upon the mirror. The principal axis is the line passing through the center of curvature and the center of the mirror surface, or its vertex. It was shown in the preceding article that this focus has definite meaning only when a small portion of the mirror surface near its vertex is used. It will be understood accordingly that all the following discussions of spherical mirrors will assume this limitation, though the figures used may for the sake of clearness overstep these limits.

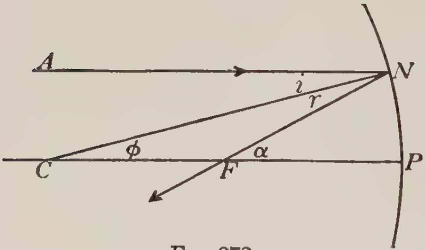


FIG. 272

Thus, in Fig. 272, the ray *AN* is *any* ray parallel to the axis of the mirror. The reflected ray *NF* makes an angle of reflection with the normal, *CN*, equal to the incident angle. Under the limitation that *N* is near to *P*, it is to be shown that *F* is midway between *C* and *P*.

$\phi = i$, since they are alternate interior angles.

$$\therefore CF = FN.$$

If N is near to P, FN is nearly equal to FP; hence CF is nearly equal to FP.

The fact that any ray parallel to the axis of the mirror passes, after reflection, through the principal focus when the point of incidence is near P, is frequently used. Thus the reflected ray may be found immediately, without the construction of angles, by drawing it through this focus F. Conversely, a point source of light placed at F will result in a parallel beam of light being sent out from that part of the mirror immediately around P. To have a parallel beam of large cross-section, a parabolic mirror is needed. Such beams, which neither diverge nor converge, conserve the energy within a beam of constant cross-section and form the common search-light beam. The source of light in this case being the hot positive electrode of a carbon arc, it is approximately a point source.

425. The Law of Spherical Mirrors.—In Fig. 273, S represents a point source of light. A divergent beam from it strikes the concave mirror mm' near the vertex P. All the rays forming this beam pass, as construction of the reflected beams would show, very approximately through a common point after reflection. This point is a focus and forms an image, I, of the source S. The law of the mirror is the relation connecting the distances of the source and its image from the mirror with the mirror constant, which is the radius of curvature r, or with the focal length f which is $\frac{1}{2}r$.

(1) The angle $\alpha = \dfrac{y}{q}$, where $q = IP$, the image distance.

(2) The angle $\theta = \alpha - i = \dfrac{y}{r}$, $r =$ the radius.

(3) The angle $\phi = \alpha - 2i = \dfrac{y}{p}$, $p = SP$, the source distance.

These relations assume that the small angles α, θ, and ϕ, subtend the same arc at the mirror. This approximation is closer, the smaller the angle.

From (2) $2\alpha - 2i = \frac{2y}{r}$.

Subtracting (1) from this, there results

(4), $\alpha - 2i = \frac{2y}{r} - \frac{y}{q}$.

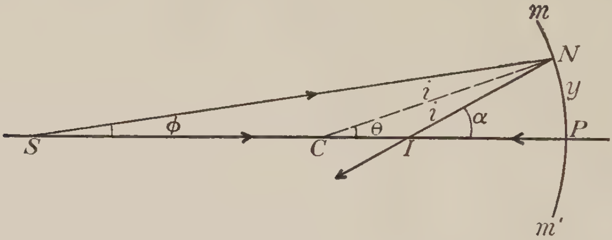


Fig. 273

Since the first members of equations (3) and (4) are equal, their second members are equal, or

$$\frac{y}{p} = \frac{2y}{r} - \frac{y}{q}.$$

From which follows

$$\frac{1}{p} = \frac{2}{r} - \frac{1}{q},$$

or

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{r};$$

or, since $f = \frac{r}{2}$,

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}.$$

In the various possible applications of this law confusion may be avoided if a convention regarding the signs of the three quantities involved is adopted. Thus, all distances are to be measured out from the mirror surface. When a distance so measured lies *in* the direction of the incident beam, the distance is considered to be a negative quantity. When the distance measured out from the mirror is opposite in direction to the incident beam, it is to be considered a positive quantity. In Fig. 273 all quantities are measured from the mirror to the left while the beam is traveling toward the right. All quantities are accordingly positive.

426. Images of Objects Formed by Spherical Mirrors.—When an object is before a spherical concave mirror, shown by the arrow in Fig. 274, each point of the object should be considered as a source from which rays proceed to the mirror.

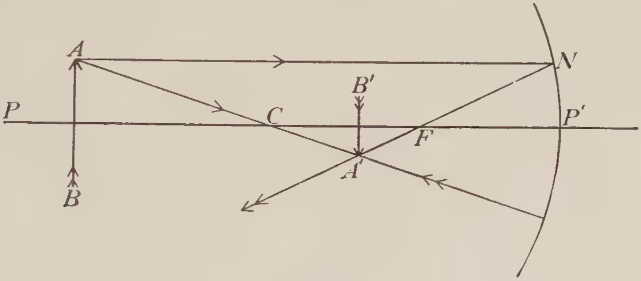


FIG. 274

When these rays after reflection pass through a common point, this point is a *real* image of the object point. When the reflected rays only appear to have passed through a common point, this point is a *virtual* image of the object point.

Since the intersection of any two rays after reflection is sufficient to locate the common point of intersection of all rays from a point source, it is natural to choose these two which are most easily traced after reflection. Thus, if the point A of the object AB is sufficiently near the axis PP', a ray AN, parallel to the axis, will pass quite accurately through the principal focus F after reflection. NFA' may be drawn immediately as one of the two rays. The ray from A passing through C will strike the mirror normally and will be reflected upon itself. This reflected ray cuts the first at A'. A' is accordingly the point where all rays starting from A intersect after reflection. It is therefore a *real* image of A. In a similar way, B' may be shown to be the image of B. The various points on the object lying between A and B have their images correspondingly placed between A' and B'. The arrow $A'B'$ is accordingly the image of the object AB. It is a real image, inverted, and smaller than the object.

If $A'B'$ were the object, two of the rays leaving the point A' would exactly retrace the paths of the two rays which have just been employed, and would make A the real image of A'. Of course, the same procedure could be used with $A'B'$ as object as was used with AB. It appears, therefore, that the positions of the object and image, when the latter is real, may be interchanged. This is due to the fact that light rays may always retrace their paths in the opposite sense.

The relation $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$ derived in §425 for a point source can also be shown to hold when the source is an object of finite size, as in the present instance.

427. Relative Sizes of Object and Image.—In Fig. 274 the similar triangles ADC and $A'D'C'$ show that the sizes of object and image vary directly as their respective distances from C.

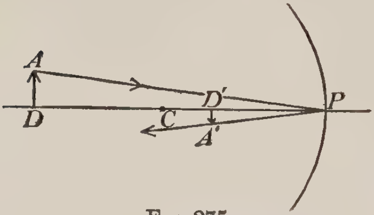


FIG. 275

In Fig. 275, the object and image of Fig. 274 are shown without the rays used to locate the image. As rays leave A in all directions, AP is a possible ray from A. As all rays from A pass through A' after reflection, this ray is reflected through A'. Since the law of reflection holds under all conditions, the angles APC and CPA' are equal, CP being a normal at the point of incidence P. APD and $A'PD'$ are similar triangles, and the sizes of the object and images are directly as their respective distances from the mirror.

PROBLEMS

1. Two plane mirrors make an angle of 60 degrees with each other. A ray of light whose plane of incidence is perpendicular to the common edge of the two mirrors strikes one mirror at an angle of 30 degrees and is reflected to the other one. Find its path after reflection.

2. Two plane mirrors meet in a common edge, which is perpendicular to the plane of incidence of a certain ray of light. This ray is reflected first from one mirror, then from the other and its final path is parallel to its original path. What is the angle between the mirrors? Prove it.

3. Show that a vertical plane mirror must have a length equal to at least one-half the height of the observer if he is to be able to see his entire figure when standing in front of it.

4. Locate the nearest six images of a point placed between two parallel plane mirrors 20 cms apart when the point is 8 cm from one mirror and 12 from the other.

5. If I measures the intensity of the illumination of a surface placed normally to a parallel beam of light, what will be the intensity when the surface is inclined at an angle of 45 degrees to the beam?

6. In the image produced by a projection lantern upon a distant screen placed normally to the axis of the beam, two adjacent vertical lines are 2 inches apart. How far apart will they be when the screen is turned 60 degrees about a vertical axis?

7. A person directly in front of a plane vertical mirror approaches it with a speed of 5 feet per second. What does the image do? If the mirror had been moved toward the person with the same speed, how would the image have moved?

8. An object 3 inches long is placed 36 inches in front of a concave mirror. The image is found to be 12 inches from the mirror. Find the radius of curvature of the mirror, and the size of the image.

9. Prove that the size of the image formed by a spherical mirror is to the size of the object as the distance of the image from the mirror is to the distance of the object from the mirror.

10. A concave mirror has a radius of curvature of 18 cms. Where must an object be placed so that the real image formed is three times as large as the object? Where will the image be?

11. A circular image of the sun is produced by its light shining through a small aperture. The image is two inches in diameter. The distance of the earth from the sun is approximately 93,000,000 miles and the diameter of the sun is 866,500 miles. How far is the screen upon which the image is produced from the aperture?

12. In problem 10 substitute the word virtual for real and solve.

CHAPTER XXXI

REFRACTION

428. The Law of Refraction.—When light strikes the surface separating two media, part of it is reflected into the first medium and a part enters the second medium. Should the surface be reflecting, as, for instance, the surface of water, a definite incident beam, of which *ON* in Fig. 276 is a ray, will suffer regular reflection along the path *NS*. Only part of the energy of the incident beam, however, travels along this path. The remaining portion enters the new medium—water, in this case—and proceeds along the path *NP* instead of along the original direction to *N'*.

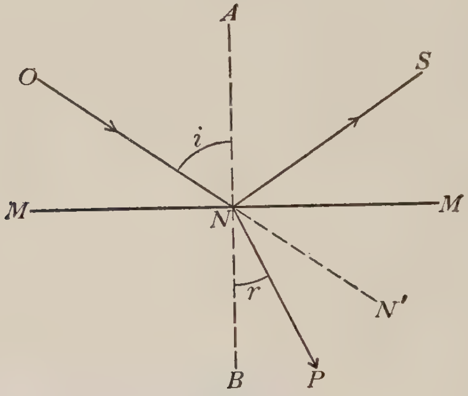


FIG. 276

There is an abrupt change in direction at the surface. This phenomenon is known as refraction. The angle *PNB*, called the angle of refraction, is distinctly smaller than the angle of incidence, *ONA*. A definite and simple relation has been found to exist between these two angles. When the angle of incidence, i, is varied in size, the angle of refraction changes, so that the ratio of the sines of the two angles remains constant. Thus $\frac{\sin i}{\sin r} = \mu$, a constant. Also the two angles lie in the same plane. This relation, discovered experimentally, may be explained in terms of both the wave theory and the corpuscular theory. The two theories, however, require opposite conditions. The wave theory requires the velocity of light to be less in the denser medium than in the rare medium, while the corpuscular theory requires it to be greater. Newton considered that a corpuscle traveling toward the surface along the line ON had a constant velocity until the surface was nearly reached. The denser medium, when the corpuscle was sufficiently close, attracted it with a force sufficient to change its direction of motion abruptly and thereby increase its resultant velocity, so that the speed in the denser medium was greater than in the rarer medium. Years later it was conclusively shown experimentally that light travels more slowly in water than in air, a fact which is in complete agreement with the wave theory.

Also it is easily understandable how a wave train can divide its energy into two trains, a reflected and a refracted train, but it is not easy to see how, in a stream of corpuscles, a part may be reflected and a part transmitted. Newton tried, but not very successfully, to endow the corpuscles with alternate fits of easy reflection and easy transmission.

The constant ratio of the sines of the angles of incidence and refraction, μ, is called the index of refraction, and varies in value with the media involved. When the incident ray is in the medium of smaller density, μ is usually greater than unity.

Should the ray proceed in the reverse direction, then the angle BNP would become the angle of incidence in the denser medium and ANO the angle of refraction in the rarer medium, and μ (water to air) would be

$$\mu_a^w = \frac{\sin r}{\sin i}.$$

This is obviously the reciprocal of the value of μ_w^a (air to water). It is customary, however, to use μ (air to water) whenever the two media are concerned, irrespective of the direction of ray; that is, to use the value of μ which is larger than unity.

429. Deduction of the Law of Refraction.—Just as it was possible in §180, Fig. 116, on Sound, to deduce the law of reflection which applies equally well to light, on the basis that sound was propagated by a wave motion, so here it should be possible by the same method to derive the law of refraction;

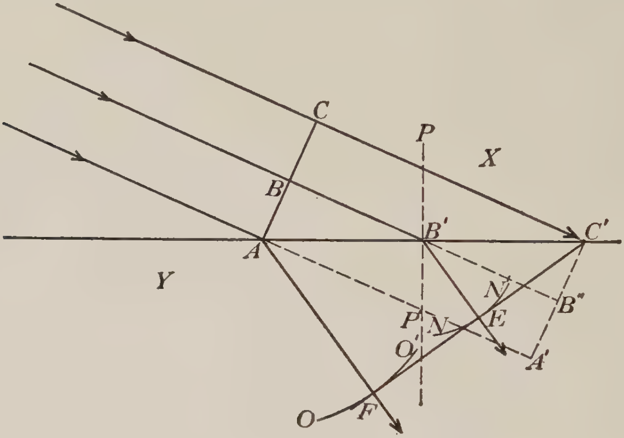


FIG. 277

$$\mu = \frac{\sin i}{\sin r} = \text{a constant.}$$

In Fig. 277 a parallel beam of light is incident upon the plane surface separating the media X and Y. ABC is a plane wave front of this parallel beam. Had the single medium X been present, the disturbance at A would have traveled to A' by the time the disturbance at C reached C', and $A'B''C'$ would have been the new wave front. The required change in direction results if it is assumed that light is propagated with smaller velocity in the medium Y which is the denser medium. For convenience in construction, this velocity is taken to be 2/3 that in X. Thus, while the disturbance at C is traveling to C' in X, that at A will travel 2/3 as far and will be somewhere on the arc OO' whose radius is 2/3 of AA'. Similarly the disturbance at B', instead of traveling to B'' by the end of this same time, will have traveled in Y a distance 2/3 of $B'B''$ and will be somewhere in the arc NN'. Similarly all the points between A and C' may be considered centers of disturbance from which secondary waves, according to Huygens' Principle, travel into Y, the distances reached being in all cases 2/3 as great as if there had been no change of medium. All the arcs thus formed will be tangent to the plane $C'EF$, which may be considered to be the refracted wave front.

The disturbance at B is propagated normally to the wave front, ABC, to the point B' from here normally to the refracted wave front to E. The angle of incidence is $BB'P$, the angle of refraction $EB'P'$. Since $CAC' = BB'P$ (sides being mutually perpendicular) and $EB'P' = AC'F$,

$$\mu = \frac{\sin i}{\sin r} = \frac{\sin CAC'}{\sin AC'F} = \frac{\dfrac{CC'}{AC'}}{\dfrac{AF}{AC'}} = \frac{CC'}{AF} = \frac{V_x t}{V_y t} = \frac{V_x}{V_y}.$$

t is the time required for the disturbance at C to travel to C'.

Thus not only is the value of μ derivable through using the wave theory, but the additional fact appears that the index of refraction is equal to the ratio of the velocities in the two media.

430. Relative and Absolute Indices of Refraction.—When the incident ray passes from a vacuum into a material medium, the value of $\frac{\sin i}{\sin r} = \mu$ is called the absolute index of refraction of the material used. Otherwise it is the relative index of the two media in contact. Where air is one of the media, the value of the relative index of air to a second material differs so slightly from the absolute index of the denser medium that the relative index is usually spoken of as the index of the denser medium.

If V equals the velocity of light in a vacuum and V_1 that in the first medium and V_2 that in the second, then the absolute indices are respectively,

$$\mu_1 = \frac{V}{V_1}, \text{ and } \mu_2 = \frac{V}{V_2}.$$

The relative index of the second with respect to the first is

$$\mu_{1,2} = \frac{V_1}{V_2} = \frac{\frac{V}{\mu_1}}{\frac{V}{\mu_2}} = \frac{\mu_2}{\mu_1}.$$

431. Intensities of Reflected and Refracted Rays.—The energy of the incident radiation is divided, in the case of regular reflection and refraction, between the reflected and refracted rays, in a manner depending upon the sizes of the angles. When the incidence is nearly normal, *i.e.*, when i and r are small, there is a minimum quantity reflected and a maximum refracted. As the value of i increases, the reflected beam gains in intensity until as i approaches 90° nearly all of the incident radiation is reflected. Thus the fraction of the sunlight reflected in the late afternoon from the surface of water is much greater than it is in midday when the incidence is nearly normal.

432. The Critical Angle.—When the incident beam is in the denser medium, as in Fig. 278, an increase in the angle of incidence results in a larger fraction of the incident energy being reflected, and a smaller fraction being refracted. In accordance with the law $\mu = \frac{\sin i}{\sin r}$, the angle of refraction increases more rapidly than the angle of incidence, so that for the incident ray SC the refracted ray grazes the surface, or the angle of refraction is 90°. That particular angle of incidence in the denser medium for which the angle of refraction in the rarer medium is 90° is called the critical angle of the two media. The surface MM' becomes a perfect reflector for all beams incident at greater angles in the denser medium.

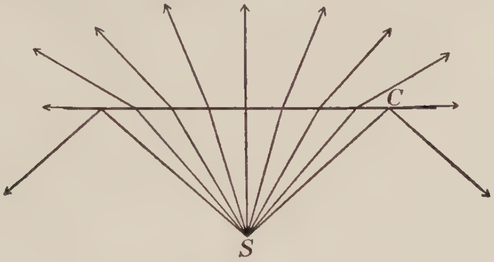


Fig. 278

A beam incident normally on the surface AB of the isosceles glass prism, right-angled at B in Fig. 279, is refracted without change of direction and strikes the surface AC at an angle of 45°. This is larger than the critical angle of air and glass. The beam is accordingly totally reflected at this surface, which becomes a perfect mirror under these conditions. The critical angle for two media is readily computed when their relative index of refraction is known. If the glass is Crown glass, its relative index with respect to air is 1.53. Thus $\mu = 1.53 = \frac{\sin i}{\sin r}$, where i and r are the angles in the air and glass respectively, irrespective of the direction in which the beam is traveling. When i becomes 90° the angle r in glass becomes the critical angle.

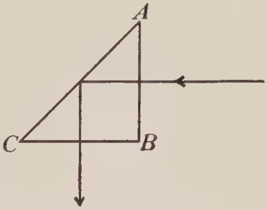


Fig. 279

$$\text{Therefore, } 1.53 = \frac{\sin 90°}{\sin r} = \frac{1}{\sin r};$$

$$\sin r = \frac{1}{1.53};$$

$$r = 40° \, 49'.$$

433. Refraction Through a Prism.—A parallel beam of light of some definite color, such as the yellow light from a sodium flame, is represented, in Fig. 280, by one of its rays PO, having an angle of incidence equal to i_1. As the prism is glass, the refracted ray is bent toward the normal, the angle of refraction being r_1. The angle of incidence at the surface AC is r_2, and the refracted angle, i_2. The ray has been deviated as a whole through the angle D which is the angle between the original direction PO and the final direction $O'Q$. This total deviation is made up of the sum of the deviations which occur at each refraction d_1 at O, and d_2 at O'. Thus

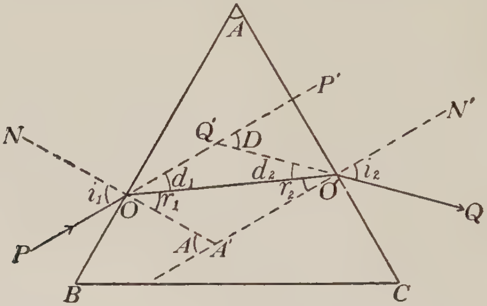


FIG. 280

$$d_1 + d_2 = D.$$

As the angles d_1 and d_2 are angles found within the prism and not directly measurable, it is desirable to express them in terms of angles readily measurable. Since the acute angle between the normals at the points O and O' equals the prism angle at A,

$$A = r_1 + r_2.$$

Therefore,

$$\begin{aligned} D &= d_1 + d_2 = i_1 - r_1 + i_2 - r_2 \\ &= i_1 + i_2 - (r_1 + r_2) = i_1 + i_2 - A. \end{aligned}$$

All of these angles may be easily determined experimentally.

434. Approximate Value of D.—When both the refracting angle of the prism and the angle of incidence are very small, the total deviation may be simply expressed in terms of μ and A.

Since the angles are small they may be used in place of their sines in

$$\mu = \frac{\sin i_1}{\sin r_1} = \frac{i_1}{r_1}, \text{ or } i_1 = \mu r_1;$$

$$\mu = \frac{\sin i_2}{\sin r_2}, \text{ or } i_2 = \mu r_2;$$

$D = i_1 + i_2 - A$ may accordingly be written:

$$\mathrm{D} = \mu r_1 + \mu r_2 - A, \quad \text{or} \quad = \mu(r_1 + r_2) - A.$$

And finally, since $r_1 + r_2 = A$,

$$\mathrm{D} = \mu A - A = (\mu - 1)A.$$

435. Index of Refraction by Means of a Prism.—If one were to vary experimentally the size of the angle of incidence i of Fig. 280, it would be found that the total deviation D of the beam by the prism would vary also. There is a particular angle of incidence, i_1, for which the deviation D is less than for any other angle of incidence. This particular value of D is called the angle of minimum deviation. It results when the angles i_1 and i_2 are equal, or when the ray passes through the prism symmetrically. Under these conditions,

$$r_1 = r_2, \quad \text{and} \quad A = r_1 + r_2 = 2r_1; \quad \text{or} \quad r_1 = \tfrac{1}{2} A.$$

From $\mathrm{D} = i_1 + i_2 - A$, there is obtained, since $i_1 = i_2$,

$$D = 2i - A,$$

$$2i = D + A,$$

$$i = \frac{D + A}{2}.$$

Therefore,

$$\mu = \frac{\sin i}{\sin r} = \frac{\sin \frac{1}{2}(D + A)}{\sin \frac{1}{2} A}.$$

Both A, the refracting angle of the prism, and D, the change in direction of the parallel beam by the prism, are easily measured with great accuracy. This is one of the important ways of determining the index of refraction of materials which can be readily used in the prism form.

436. Lenses.—The function of lenses, as well as of spherical mirrors, is to change the shape of incident wave fronts. Thus in §424, if a source of light is placed at F, as in Fig. 272, the divergent beam incident upon the spherical mirror will have the convex wave fronts changed to plane wave fronts in the parallel reflected beam. In the case of convex lenses, the tendency is to retard the portion of a wave going through the center of the lens with respect to that portion passing through the lens near the edges. Thus divergent waves are rendered less diverging, or even concave, and plane waves are changed to concave waves. In Fig. 281, spherical waves diverge from the source S. Upon reaching the lens L, the portions of the waves intercepted are shown changed from convex waves into concave waves. As light travels more slowly in glass than in air, the greatest retardation is where the glass is thickest. Thus the part of the wave passing through the middle of the lens is much more retarded than that passing near the edges. In this instance, this retardation is sufficient to place the central portion of the wave behind its edges, and thus to create a concave wave which

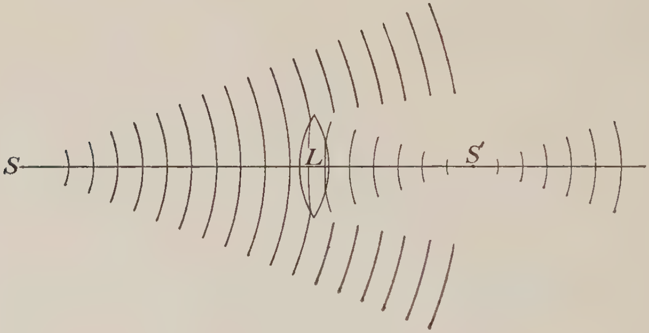


Fig. 281

advances to a point focus at S', from which the energy again travels forward by means of a convex wave train. For an eye placed anywhere in this train the origin is S'. S' is accordingly a real image of S. Any lens thicker in the center than the edges tends to retard the central portion of the waves passing through it with respect to their edges, and may accordingly be considered as a converging lens. Lenses thinner at the center than at the edges are diverging lenses.

The construction in Fig. 281 is a rough approximation only of the action of such a lens as L. L would not in fact, for one thing, produce concave waves which were really spherical. The portion of the waves passing through the lens near its edges would have the greatest curvature and would accordingly come to a focus at a point nearer the lens than the portion passing through the central portion of the lens. This effect is one type of spherical aberration.

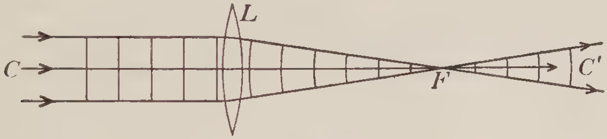


FIG. 282

An exact treatment of lenses is a complicated problem and is beyond the scope of this text. By assuming that the lenses are all very thin in comparison to their other dimensions, and by using only their central portions, great simplification of treatment is possible. The results thus obtained, while only approximations, will serve to demonstrate many of the fundamental properties of lenses. Thus L, in Fig. 282, represents a thin lens upon which a beam of parallel rays is incident. The lens transforms the plane wave front of this beam into a concave front, which draws into a point at F. F is called the *principal focus* of the lens, since the incident radiation is parallel to the axis of the lens, CC'. The distance of this focus from the lens depends upon the curvature of the two faces of the lens and upon the kind of glass used, that is, upon its index of refraction. This focus may be determined experimentally with considerable accuracy by measuring directly the distance of the image of a distant object from the lens. In the following demonstrations the position of F is assumed to be known.

437. Images Formed by Lenses.—Light diverging from any point of an object and passing through the ideal simple lens either converges to a point or appears to come from a point. This point is in either case an image of the point source. In determining the position of this image, the method used in the case of spherical mirrors is most satisfactory. It is only necessary to find where two rays starting from a point source meet, or the point from which they appear to diverge after passing through the lens. Since all rays starting from the same point source come to a common focus, this image point may be located by the intersection of any two refracted rays. Accordingly, those two are chosen whose directions after passing through the lens are most easily determined. Thus let AB, Fig. 283, be any object whose image is to be formed by the simple lens L. To locate the image of the point A, a ray from A parallel to the axis CC' is one of those chosen, since this ray will pass through the principal focus F after passing the lens. Assuming the location of this focus at F to be known, this refracted ray proceeds along the line FA'. A second ray from A passing through the point O suffers no bending, since the tangents to the lens surfaces where it enters and leaves the lens are parallel. This ray in effect passes through a plane parallel glass plate and experiences only a slight lateral displacement. This may be

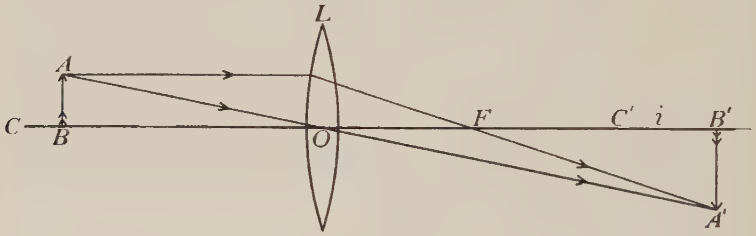


FIG. 283

neglected, since the lens is assumed to be of negligible thickness. Since two rays starting from A meet at A' after passing through the lens, A' is a real image of A. Waves diverging from A are changed by the lens to converging waves which draw in to A' as center. After passing this point they become diverging waves, with A' as a new source.

In a similar way the images of all points in succession from A to B of the object may be found. It is evident, from the construction used, that they will lie in symmetrical positions between A' and B'; or $A'B'$ is a real image of AB. The similar triangles ABO and $A'B'O$ show that the sizes of image and object are as their distances from the lens center O. Furthermore, the image is inverted.

In the construction used, not only was F assumed to be known but the fact that refraction occurs at both faces of the lens was neglected. This last approximation is justified, since our lenses are supposed to be of negligible thickness, that is, much thinner than the figure shows.

438. Virtual Images.—Where the object lies nearer to the lens than the focal distance, as in Fig. 284, the same method of construction leads to the formation of a virtual image $A'B'$, which is erect. The sizes of the image and object are as their distances from the lens.

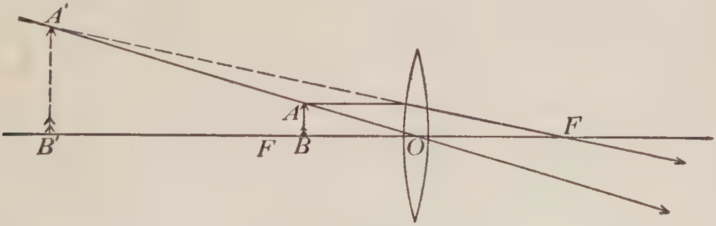


FIG. 284

The waves from A are so convex when they come to the lens that the lens cannot change them into concave waves; instead, it transmits them as convex waves having a smaller curvature. To an eye receiving these altered waves, the source appears to be at A'.

439. The Law of Lenses.—It is desirable to be able to express the distances of the object and image from a lens in terms of the lens constants. These are the radii of the two spherical faces and the index of refraction of the glass, μ. In Fig. 285, PP' represents an ideal simple lens of negligible thickness. The source is S, and the point of incidence of the ray SP at P is to be considered very close to the lens axis C_1C_2, so that all the angles are very small.

For any ray the upper half of the lens PO is in effect a very thin prism, and the deviation found in §434 for a ray passing through a thin prism will apply here. Thus $D = (\mu - 1)A$, where A is the refracting angle of the thin prism. Here A is the small angle formed by the tangents to the spherical surfaces at P, and is accordingly equal to the acute angle between the radii drawn to this point.

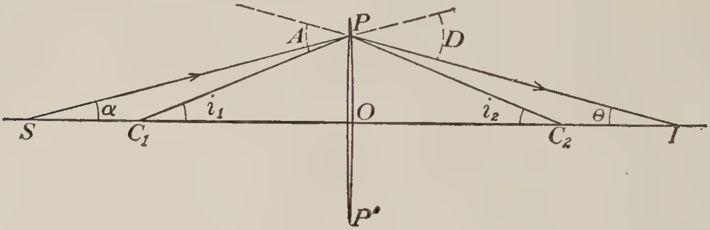


FIG. 285

From the figure,

$$A = i_1 + i_2 = \frac{PO}{r_1} + \frac{PO}{r_2},$$

where r_1 = radius C_1P and r_2 = radius C_2P.

$$\text{Therefore } D = (\mu - 1)\,A = (\mu - 1)\,PO\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

Also from the figure,

$$D = \alpha + \theta = \frac{PO}{SO} + \frac{PO}{IO}.$$

Since SO is the object or source distance from the lens, it is written equal to p, or $SO = p$. Similarly, $IO = q$, or the image distance.

$$D = PO\left(\frac{1}{p} + \frac{1}{q}\right).$$

Placing the two expressions for D equal,

$$PO\left(\frac{1}{p} + \frac{1}{q}\right) = (\mu - 1)PO\left(\frac{1}{r_1} + \frac{1}{r_2}\right);$$

$$\frac{1}{p} + \frac{1}{q} = (\mu - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

The second member of this equation is made up of the lens constants, μ, r_1 and r_2. The equation gives the relation sought.

Since all the quantities in the second member are constants, the second member itself is a constant, and

$$\frac{1}{p} + \frac{1}{q} = \text{constant.}$$

The value of this constant could be determined once for all if q were known for a given value of p. A general value for the second member can, however, be readily determined. Thus, when S is at a great distance, $\frac{1}{p}$ becomes very small and may be neglected. In this case,

$$\frac{1}{q} = (\mu - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

Since the source is in effect at infinity, q is the distance of the principal focus F from the lens, and this particular value of q is f. Therefore,

$$\frac{1}{f} = (\mu - 1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right).$$

Accordingly, $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$ is a second form of the lens law.

In applying this formula the distances of virtual sources and images are considered negative, and their numerical values are inserted in the equation with negative signs. Also, diverging lenses have principal foci which are virtual. In their case the numerical value of f is to be given a negative sign. The radius of curvature of a concave surface is to be used with a negative sign.

PROBLEMS

1. If a ray of light makes an angle of 30 degrees with the surface of water what deviation will it suffer as it enters the water? $\mu = 1.333$.

2. The index of refraction of turpentine for yellow light is 1.48. Find the critical angle.

3. A 60-degree prism of flint glass having an index of refraction of 1.6 will produce what minimum deviation?

4. A plano-concave lens of flint glass is to have a focal length of 100 cms. Find its radius of curvature. Index of refraction of the glass = 1.6.

5. An object and a screen are 250 cms apart. Where must a lens having a focal length of 40 cms be placed to produce a clear image on the screen? Show that there are two solutions and find the relative size of image and object in each case.

6. Draw a double convex lens to a good sized scale. Draw two lines to represent rays of light parallel to the principal axis, one being about twice as far from the axis as the other. Trace these two rays through the lens, making necessary constructions to produce accurate work. Do these rays show the effect of spherical aberration?

7. Determine the critical angle for a medium in which the velocity of light is 110,000 miles per second.

8. If the real image produced by a convex lens has the same size as the object, find the distance between the image and the object in terms of the principal focal length of the lens.

9. A fish pole is held partially under water and at an angle of about 30 degrees to the horizontal. Make a drawing showing the rays of light by which the lower end is seen by the person holding the upper end. Show also the figure of the pole as it would appear. Index of refraction of water = 1.33.

CHAPTER XXXII

SOME SIMPLE OPTICAL INSTRUMENTS

440. The Eye.—The lenses of the eye are thick and, particularly in the case of the crystalline lens, $BB'C$ in Fig. 286, not of uniform structure. They cannot therefore be treated as simple lenses. For present purposes, however, the ideas developed by the aid of simple lenses are quite sufficient. AA' represents the outer surface of the first transparent layer of the eye, the cornea. The chief refractions of the eye occur at A and at the first and second surfaces of the crystalline lens at B and C. With a source at infinity, the normal eye will focus the beam on the retina RR, on whose surface the principal focus F, should lie.

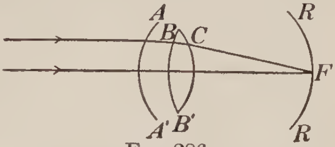


FIG. 286

For nearer sources the images would lie farther from the refracting lens, if the crystalline lens did not possess the power of adjustment or accommodation. By muscular contraction this lens may be made more converging, so that waves starting as near as 25 cm from the eye may still be made to converge upon the retina. The image on the retina of a given object increases in size as the object approaches the eye. With a larger image there is a corresponding increase in the detail which can be perceived, so that when detail is desired, objects are brought up to this minimum distance. In reading, the book is usually held at this distance from the eyes. The fact that all of the images on the retina are inverted causes no difficulty, since experience has taught their correct interpretation.

441. Some Defects of the Eye.—1. *Farsightedness.* When a person sees distant objects clearly but has difficulty in seeing those nearby, he is farsighted. The crystalline lens of his eye cannot be made sufficiently convergent to bring the images of near objects to focus on the retina; instead, the images fall behind the retina. By placing spectacle lenses which are of the right convergence before his eyes, these images may be brought forward sufficiently to lie upon the retina.

2. *Nearsightedness.* When objects nearby can be seen distinctly, while objects at a distance cannot be seen, a person is nearsighted. The lenses of the eyes are too converging; that is, the surfaces of the lenses have too great curvature. Glasses with lenses which are diverging in the proper degree are required to correct this defect.

3. *Astigmatism.* When vertical and horizontal lines cannot be seen equally well at the same time it usually means that the cornea has a greater curvature in one of its meridians than in another. This defect is corrected by cylindrical lenses.

442. Apparent Size of an Object.—*Magnification.* The apparent size of an object is judged by the angle it subtends at the eye. This *visual angle* is measured by dividing the size of the object by its distance away. Thus, in Fig. 287, $\frac{AB}{CO}$ measures the visual angle. This is obviously equal to the length of the image divided by its distance OC', or $\frac{A'B'}{OC'}$.

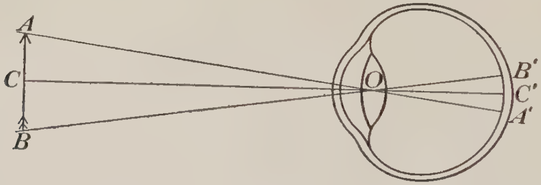


Fig. 287

To see the details of an object more clearly, it is necessary to increase the size of the image $A'B'$ on the retina. Since the distance OC' is fixed, this increase in size of retinal image $A'B'$ means increase in size of the visual angle. This is done, with a given object AB, by decreasing OC. It has already been stated that this distance cannot be decreased beyond 25 cm for the average eye. There are optical means of decreasing this distance still further and thus increasing the visual angle and the apparent size. The ratio of the apparent size of the object, when viewed through the optical instrument and when viewed directly, is called the magnification of the optical device. Thus

$$\text{Magnification} = \frac{\text{apparent size through instrument}}{\text{apparent size without instrument}}.$$

443. Simple Magnifying Glass or Microscope.—When a lens of short focal length, as L of Fig. 288, is held directly in front of the eye and an object AB is placed before the lens just inside of the principal focus F, the eye sees the virtual image $A'B'$. By causing the object AB to approach F, this image rapidly moves away and would be at infinity if AB should reach F. However distant $A'B'$ may be, its visual angle does not change appreciably and may be placed equal to AB/f, where f is the focal length OF of the lens L. Since $A'B'$ is very distant, the waves apparently coming from it are sensibly plane waves and are readily brought to focus by the normal eye on the retina. Accordingly, since the eye with the lens before it sees the object subtending the angle AB/f, the apparent size through the instrument $= \dfrac{AB}{f}$. The apparent size without instrument when viewed directly by the eye is $\dfrac{AB}{25}$, since 25 cm is the distance of greatest distinctness of vision. Therefore the magnification $= \dfrac{AB}{f} \div \dfrac{AB}{25} = \dfrac{25}{f}$.

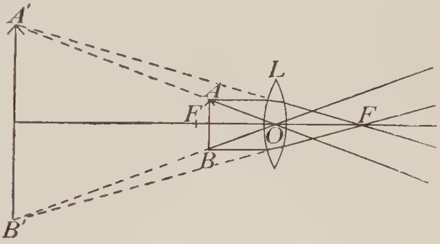


FIG. 288

Thus, if the lens L has a focal length f equal to 1 cm, the magnification produced by placing the lens immediately before the eye is 25. Lenses with shorter focal lengths may still be used with resulting greater magnification, but such short focal lengths are not usually convenient.

If, instead of placing the object at the focus of the lens, one places it at such a distance, p, as to produce a virtual image at 25 cms, the normal reading distance, the magnification is somewhat increased. p now takes the place of f in the expression for magnification. From the lens formula,

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{f}.$$

Since the image $A'B'$ is virtual, the image distance q is -25, so that our expression becomes

$$\frac{1}{p}=\frac{1}{f}+\frac{1}{25},$$

and

$$\frac{25}{p}=\frac{25}{f}+1.$$

444. Astronomical Telescope.—This instrument, shown in Fig. 289, has for its optical parts as large an objective L as possible, with a long focal length F, and small lens l of short focal length as eyepiece. For this discussion the lenses will be treated as simple. They are, however, compound lenses in good instruments.

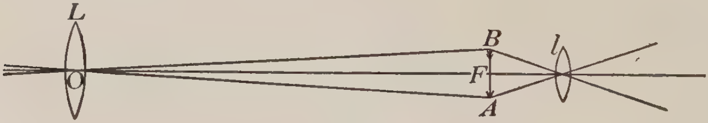


FIG. 289

Since telescopic objects are distant objects their images are real images, many times smaller than the objects themselves, and lie inverted at the principal focus of the lens, as at AB.

The lines OB and OA, coming from the extremities of the distant object, pass through the lens center O, and determine the size of the image AB, after the manner developed in Figs. 283 and 284, where it is seen that the point source, lens center, and image point all lie on a straight line.

The object, looked at directly with the eye, subtends the same angle as it subtends at O in the lens L. That angle is numerically equal to $\frac{AB}{F}$. Thus, while neither the size of the object itself nor its distance is known, its visual angle or apparent size is readily determined. Similarly, lines from the extremities of the image AB through the center of lens l will determine the visual angle the image subtends at the eyepiece, which is used as a simple microscope. The small image AB is magnified by the eye piece l, its apparent size being AB/f, where f is the focal length of lens l, as shown in §443. Since through the telescope the astronomical object appears to subtend the angle AB/f, and without the telescope the angle AB/F, the magnification is equal to

$$\frac{AB}{f} \div \frac{AB}{F} = \frac{F}{f}.$$

445. The Compound Microscope.—When an object which is to be highly magnified is placed just beyond the principal focus of a short focused lens O, Fig. 290, there is formed a real, magnified and inverted image of the object. The ratio of their sizes is given by the equation $\frac{A'B'}{AB} = \frac{L}{F}$. This image is viewed with a simple microscope as eyepiece, as described in §443. The magnification is given by the ratio of the two visual angles,

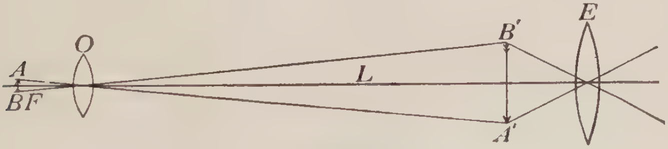


FIG. 290

$$\frac{A'B'}{f} \div \frac{AB}{25} = \frac{A'B'}{AB} \cdot \frac{25}{f} = \frac{25L}{fF}.$$

In this expression L is approximately equal to the length of the microscope tube.

The similarity of idea in the construction of telescope and microscope is evident. There are short-focus telescopes and long-focus microscopes which serve to bridge the gap between the two instruments.

446. Spectrometer.—When a source of light of some definite color, such as yellow, is placed before the slit S of the spectrometer, Fig. 291, the beam passing through the slit diverges and fills the lens C at the farther end of the tube. This lens is adjusted so as to have the slit S at its principal focus, with the result that the beam, as it leaves C, consists of parallel rays.

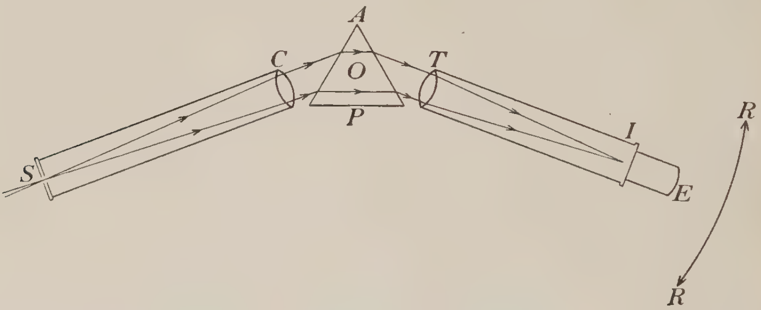


Fig. 291

This tube SC, containing slit and lens, is called the collimator and has for its function the production of a parallel beam. This parallel beam passes through the prism AP, and is deviated, as shown in the figure. The telescope TE, which can rotate about an axis O in the directions RR, is turned to receive the refracted beam. Its lens T brings the beam to a focus at I, its principal focus. There is at I a real image of the slit S. The figure is a horizontal cross-section of the instrument, and shows the width of the slit S. The length of the slit may be a centimeter and is normal to the plane of the paper. Similarly, P is a cross-section of the prism. Its height, which may be 2 cms, is normal to the paper. Accordingly the image at I is also a line whose length is normal to the paper. If the two lenses C and T have equal focal lengths, as is usual, the image at I is of exactly the same size as the slit S.

The prism is mounted upon a table which rotates about the same axis as the telescope. By rotating the prism about this axis, the position giving minimum deviation can be found. The spectrometer is an instrument by which D, the angle of minimum deviation, and the angle A of the prism can be accurately measured. As was shown in §435, these quantities may be used to determine the index of refraction of the material of which the prism is made. The experimental method used in making these measurements may be found in any laboratory manual of physics.

This instrument is described because in some of its various forms it gives important information in many fields of science.

PROBLEMS

1. A person who is far-sighted wishes to obtain a simple magnifying glass that will make an object look three times as large as it looks to him at a distance of 18 inches. The lens is to be double convex and made of glass whose index of refraction is approximately 1.5. What should be the radii of curvature?

2. A near-sighted person can read with comfort a paper 5 inches from his eyes. What kind of glasses should he have to be able to read at a distance of 12 inches? What is their focal length?

3. The distance from the optical center of the objective of a telescope to the optical center of the eye piece is 63 cms. Its magnifying power is 20. Find the focal length of each of the two lenses.

4. A convex lens of focal length 60 cms produces an image of a lamp placed 120 cms from it. A second convex lens of 30 cms focal length is placed 40 cms from the first lens on the side opposite the lamp. Compute the position of the image formed by the two lenses.

5. A camera focused to take a picture of the moon has its lens at a distance of 25 cms from the plate. How much must the lens be moved to take an object 10 meters away?

6. When a reading glass in the form of a large convex lens of about 12 cms focal length is used at a distance of 8 cms above the printed page, where is the image?

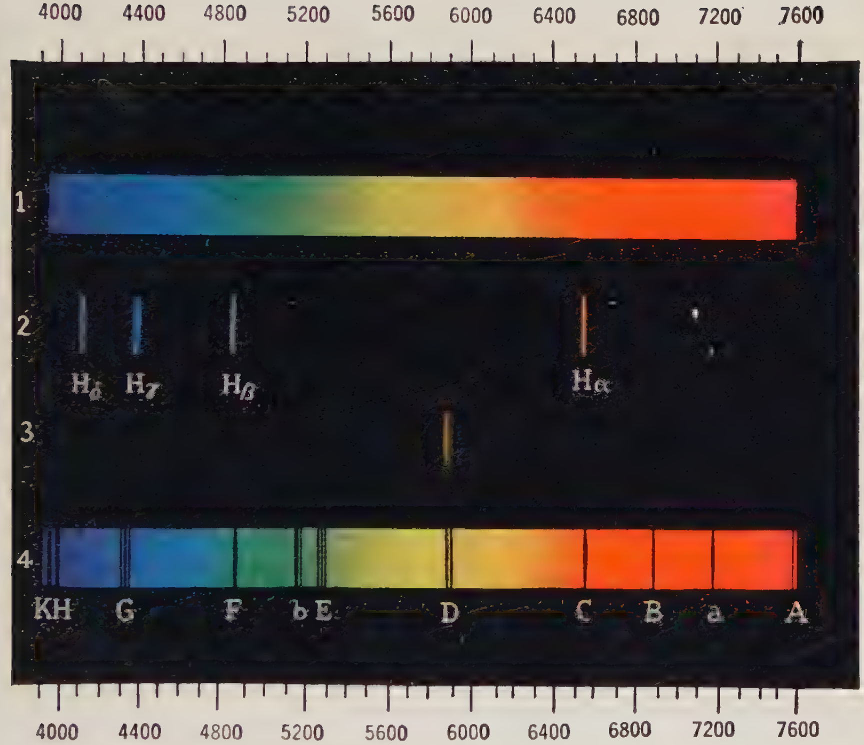
CHAPTER XXXIII

THE SPECTRUM AND COLOR

447. The Spectrum.—When there is placed before the slit of a spectrometer, as shown in Fig. 291, a glass tube containing hydrogen gas under reduced pressure which has been made luminous by the passage of electrical discharges through it, then, instead of the single yellow image of the slit which was obtained in the previous experiment when yellow light was used, there appear four images of the slit—a red, a blue, and two violet images. Of these images, the red image is deviated least from the direction of the beam incident upon the prism, and the violet images the most. Several inferences may be drawn from these observations:

1. The present source is complex in that it emits at least four colors instead of one, as in our first source.
2. Since there is a single angle of incidence upon the prism, and four angles of refraction, there are four indices of refraction, one for each color.
3. Since the index of refraction is the ratio between the velocities of the light in air and in glass, there are also four velocities in the glass, one characteristic of each color.
4. As the bending comes from a decrease in velocity, the change is greatest for the violet, which is bent the most. The velocity of the red light is greater than that of the blue, which is greater than that of the violet.

These conclusions are based upon the fact that all have essentially the same velocity in air. This succession of colored images of the slit is called the spectrum of hydrogen gas. This spectrum is shown in line 2 of Plate I. Had a mercury vapor lamp been used, there would have appeared as the spectrum



1. A continuous spectrum
2. The visible part of the Hydrogen spectrum
3. The visible part of the Sodium spectrum
4. The solar spectrum showing the lettered Fraunhofer lines

PLATE I

(From Duncan's *Astronomy*, published by Harper & Brothers)

of mercury two yellow lines quite close together, a green line and a violet line, each line being again a colored image of the slit. When the white-hot carbon of an arc is the source, there appears as the spectrum a continuous band of color, starting with red and changing by imperceptible steps through the succession of colors—orange, yellow, green, blue and violet. This is shown in the upper spectrum of Plate I. Considering that here also a succession of colored images of the slit is being dealt with, it appears that there must be an indefinite number of colors in the radiation from this source, since there are no gaps indicating that certain colors are absent. Each color of this spectrum would be characterized by a particular index of refraction and velocity of propagation. This last spectrum is a *continuous spectrum*, while those of hydrogen and mercury gases are *bright line spectra*.

448. Color of Opaque Objects.—When the continuous spectrum from the arc is projected upon a white screen, as may be done by replacing the telescope of Fig. 291 by a single lens near the prism, the succession of colors which was first seen by looking through the telescope appears upon the screen and can be seen from any angle. The white screen has the property of reflecting equally all colors, since they all appear to have the same intensities whether viewed directly by looking through the telescope or indirectly after reflection from the screen.

When, however, flowers of different colors, or colored cloths, are moved through the spectrum from one end to the other, it at once becomes evident why they are colored. Thus, a red flower or a red cloth appears red when held in the red end of the spectrum, and black when in any other color. Each can reflect incident red light, but no other color. Similarly, a blue flower may appear black when any other color than blue is falling upon it. Colored objects are thus colored because they have the power of reflecting their characteristic colors. Light of other colors falling on them is absorbed, that is, changed into heat.

When the source of light does not emit light of the color which certain objects reflect, those objects have to appear black. Thus light from the mercury lamp showed only yellow, green and violet radiation. A room lighted by such lamps would show all red objects as black objects. It is very important, therefore, that artificial lights emit those colors contained in sunlight and with the same relative intensities, if colored objects are to appear the same by artificial light as by daylight. The spectrum of the present-day tungsten lamp is very similar to that of the sunlight, except that relatively the yellow, orange, and red end of the lamp spectrum is brighter than its blue end. These colors in objects are accordingly emphasized in artificial light, while the greens, blues and violets are somewhat suppressed.

449. Colors of Transparent Objects.—Transparent objects which appear to be colored owe this property to the light which they transmit. If a piece of glass appears to be green, it is because the green from the white light that illuminates it is freely transmitted, while most of the other colors are absorbed.

The bulbs of daylight lamps which appear bluish transmit the blue end of the spectrum without absorption but do absorb enough from the red end to reduce the relative excess of the red, orange, and yellow light emitted from tungsten filaments. The spectral distribution of color in the daylight lamp is thus made essentially that of sunlight.

450. Mixtures of Colored Lights.—The spectrum is made up of a succession of colored images of the slit. If the spectrum is short and the slit wide, these images overlap and the spectrum is an impure spectrum in the sense that if the light found at any narrow part of it were sent through a second spectrometer it would be found to consist of several colors, depending upon the amount of overlapping. Increasing the length of the spectrum or decreasing the width of the slit tends to prevent this overlapping of images and the mixing of the adjacent colors of the spectrum. The more successfully this overlapping is prevented, the more nearly the spectrum is a pure spectrum.

If two small mirrors were mounted in the spectrum before the screen in such a way as to reflect blue light from one and from the other yellow light, and both beams were directed to a common spot on a white screen, it would be found that such tints of yellow and blue could be chosen that their combination would produce white light. Unlike the ear, which can analyze a complex sound into its components, the eye cannot tell that this spot is white because yellow and blue light are in combination. Two colored lights which, combined in this way, produce white light in the physiological sense, are called complementary colors.

All known colors may be duplicated by combining in the above fashion the three colors—red, green and violet. They have been called for this reason the primary colors. They are no more primary than the other colors of the spectrum, from the physicist's point of view. The question as to why all colors can be matched by combinations of these three is one which is not yet solved. It is a problem for collaborative research, involving physics, physiology and psychology. The same is true of many of the phenomena of sight.

451. The Mixture of Pigments.—Colored objects always reflect or transmit several or even many colors. The colors of paints are good illustrations of this. Since mixing yellow and blue light produces white light, it will be interesting to see why mixing yellow and blue paints gives green paint. The yellow paint reflects not only yellow but also perceptible quantities of the adjacent spectral colors, orange and green; the blue paint reflects green and violet in addition to the blue. When the two are mixed, the only one of the five colors involved—orange, yellow, green, blue and violet—which is not quite completely absorbed by one or the other paints is the green. Green produced this way cannot be a very bright color.

452. Dispersion.—Referring again to our bright line spectra, assume that a substance in gaseous form is producing a spectrum of a number of lines, and that one of these lies at the red end of the spectrum, one in the yellow region or middle part, and one at the violet end. The red and violet lines serve to limit the extent of the spectrum, and the yellow line to locate its center. Two prisms, each similar to that of Fig. 292, having the same refracting angles A but made of different kinds of glass, produce spectra containing the same lines, but the distribution of these lines in the two spectra is not the same. The differences to be considered here are mainly of three kinds: the angular opening or spread of the spectrum measured by the angle α, the deviation of the spectrum as a whole from the original direction as measured by δ_Y, and the relative positions of the various lines which make up the two spectra; thus the yellow line Y may be relatively nearer V and farther from R in one spectrum than in the other.

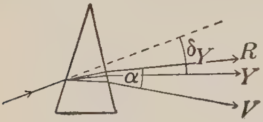


FIG. 292

For simplicity, prisms with small refracting angles A have been chosen. The deviation is then given by $D = (\mu - 1)\ A$, as derived in §434. Let d_R, d_Y, and d_V represent the deviations of the red, yellow and violet lines, respectively, from the original direction as shown by the dotted line. Then

$$d_R = (\mu_R - 1)A,\ d_Y = (\mu_Y - 1)A,\ d_V = (\mu_V - 1)A,$$

$$\alpha = d_V - d_R = (\mu_V - 1)A - (\mu_R - 1)A = (\mu_V - \mu_R)A.$$

Thus the length of the spectrum is proportional to $\mu_V - \mu_R$; or the difference between the indices of refraction of the prism for the extreme rays measures the ability of the prism to open up the spectrum. This difference is called the dispersion of the substance. It measures the angular opening of the spectrum per unit refracting angle of the prism. This quantity is different for different kinds of glass.

Another quantity is the average deviation of the spectrum as a whole. Since the yellow is often the brightest part of the spectrum, its deviation is generally taken as a measure of the deviation of the entire spectrum. Thus $d_Y = (\mu_Y - 1)A$ equals the mean deviation of the whole spectrum.

The expression, $\dfrac{\mu_V - \mu_R}{\mu_Y - 1}$, is called the dispersive power of an optical substance. Since the numerator measures the angular opening of the spectrum per unit refracting angle and the denominator is the mean deviation of the spectrum per unit refracting angle, the quotient must measure the angular opening of the spectrum per unit deviation of the spectrum. This expression measures a very valuable property of optical glasses; and it is largely because it has become possible to make glasses having widely differing values of this expression that such advances have recently been made in optical instruments.

453. Achromatic Prisms and Lenses.—*A* and *A′* in Fig. 293 are two prisms made of such different glasses and refracting angles that, while the angular opening of the spectrum is the same in both cases, the mean deviation of the spectrum for prism *A* is greater than for prism *A′*; that is, d_1 is greater than d_2. If the two prisms are placed together, as shown in the last figure, the combination results in no separation of the incident beam into a spectrum. This is because the angular openings of the spectra are the same, while the order of the colors is now reversed in space. It will be noted that for simplicity no attempt is made to show dispersion within the prisms in the last figure.

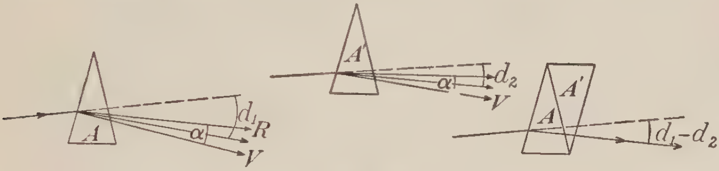


FIG. 293

The undispersed beam is deviated downward an amount d_1 by prism *A*, and upward an amount d_2 by prism *A′*, leaving a resultant deviation toward the base of *A* equal to $d_1 - d_2$. Thus it is possible to make two prisms and so mount them that they will bend a beam of light without resolving it into its component colors, that is, without producing a spectrum.

Since the relative positions of the different colors of the spectrum produced by prisms of different glass are not the same, even when the spectra have the same length, there is not complete suppression of color by this method.

454. Direct Vision Spectroscope.—It is frequently convenient to have a spectroscope which is more readily carried about than the one shown in Fig. 291. It is possible to design a spectroscope in which a single short straight tube will contain all the essential parts of a spectrometer. The greatest change is in the substitution of a direct vision prism for the one in Fig. 291, which deviates the entire spectrum. Fig. 294 shows two prisms A and A', having such angles and indices of refraction that both spectra suffer the same deviation, d. Prism A gives a spectrum of small angular opening, α_1, while A' gives a relatively large angular opening, α_2. The two are combined as in the third figure, where no attempt is made to trace the rays within the combination. There is no resulting deviation of either spectrum; the deviation of the yellow line would be downward by the angle d_y through the action of A and upward by the same angle d_y through the action of A'. The ray Y proceeds through without deviation; therefore,

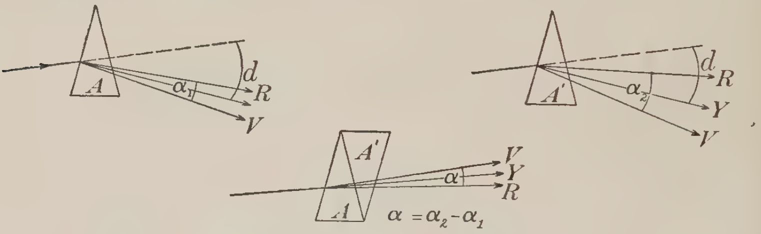


FIG. 294

$d_y = A(\mu_Y - 1) = A'(\mu'_Y - 1)$ expresses this condition.

Since A' produces a spectrum of angular opening α_2, and A one of smaller opening α_1, the opening α, when the two prisms are dispersing in opposite directions, will be $\alpha = \alpha_2 - \alpha_1$. This is a shorter spectrum than is usually obtained by the use of a single prism as in Fig. 291. The actual instrument may contain a combination of several prisms instead of two.

455. Chromatic Aberration and Achromatic Lenses.—A lens may be considered to be made up of two thin prisms placed base to base, with curved instead of plane faces. Each half must have the general properties of a prism. Each half should resolve incident radiation into a spectrum. Reference to earlier discussions of lenses will show that this property of lenses was not then considered. Let S in Fig. 295 be a source of complete radiation. The resolution of this radiation by the lens into its component parts is due to the fact that the red rays are deviated less by refraction than are the violet rays, and they come to a focus at a point farther from the lens.

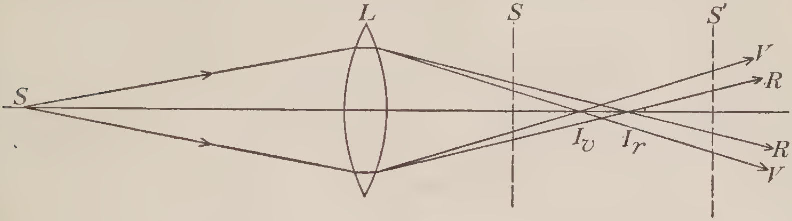


Fig. 295

The foci for red and violet rays are indicated in Fig. 295 by the points I_r and I_v. The other colors would have foci lying on the lens axis between these two foci. For a source which is not monochromatic, there is no definite focus for radiation sent out from a point, but rather there are as many foci as there are component parts in the radiation. Screens placed at S and S' should show illuminated spots bordered with red at S and violet at S'.

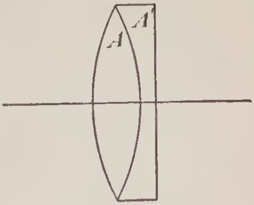


Fig. 296

Fig. 296 in its upper half is similar in its optical properties to the combined prisms A and A', as shown in the third part of Fig. 293, where there is a deviation produced by the combination, but no resolution into colors. By proper choice of glasses and curvature of surfaces, the lens A' may be made to recombine any spectrum produced by A while only partly neutralizing the deviation caused by A. The combination will accordingly act as a converging lens without production of color. Here also, as in the case of the prisms, the fact that different glasses proportion differently the space occupied by the various colors in spectra of the same length prevents such a combination of two glasses from wholly suppressing the appearance of color. The finest achromatic lenses are made of several lenses.

456. The Velocity of Light.—While light, in all that has preceded, has been assumed to travel with a definite velocity, no statement has been made concerning the magnitude of this velocity. Fizeau, in 1849, was the first to measure directly the velocity of light by determining the time required for a beam of light to travel to a distant point and back. At one point in its path near the light source this beam passes between the teeth of a wheel which can be rotated at any desired speed. The reflecting mirror at the distant point is adjusted so that, when the wheel is stationary, the returning beam passes between two of the teeth, and is focused in a telescope. As the wheel is rotated, a speed is reached such that the returning light is blocked by a tooth and the image is eclipsed. Further increase of speed causes the image to reappear when the light on returning again finds an opening in front of the telescope. This means that the time consumed by the light in going to the mirror and back equals the time required for the wheel to turn the distance between two adjacent openings between teeth. This time t, being known from the speed of rotation of the wheel, the velocity of light V is given by the equation

$$V = \frac{2L}{t},$$

where L is the distance from the rotating wheel to the distant mirror. Fizeau's measurement gave for the velocity of light the value 315,000 kilometers per second. Later determinations by methods which were modifications of Fizeau's have given a value very close to 300,000 kilometers per second.

PROBLEMS

In § 452 the spread of the spectrum is shown to be $(\mu_v - \mu_r)$ A where *V* and *R* signify the violet and red rays determining the limits of the spectrum. In practice, the Fraunhofer lines B and H, shown in Line 4 of Plate I, are frequently taken as spectral limits and the D line of sodium replaces the yellow ray of Figure 292. The following Indices of Refraction will be useful:

Fraunhofer lines	A	B	C	D	E	F	G	H
Wave lengths in Angstroms	7594	6867	6563	5890	5270	4861	4308	3968
Flint Glass		1.6127		1.6193				1.6527
Crown Glass		1.5301		1.5339				1.5509
Alcohol	1.359	1.360	1.361	1.363	1.365	1.367	1.371	1.374
Water	1.329	1.331	1.332	1.334	1.336	1.338	1.341	1.344

1. What is the angular opening, α, between the B and H lines of the spectrum produced by a flint glass prism of 6 degrees refracting angle? What is the dispersive power of this prism in this same spectral region?

2. What are the dispersive powers of alcohol and water for the spectral region between the C and H lines?

3. Find the angle of a flint glass prism that will achromatize the region from B to H in a crown glass prism whose angle is 4 degrees. What deviation will the combination produce for light of the D line?

4. In making a direct-vision prism combination using crown and flint glass, a crown glass prism of 18 degrees was used. What should be the refracting angle of the flint glass prism and what will be the angular opening between the B and H lines?

5. With a crown glass prism having a refracting angle of 60 degrees what will be the emergent angles for the rays B and H, when their common angle of incidence is 50 degrees?

6. How many vibrations per second correspond to a wave length of 5000 A? How many wave lengths of this radiation make a cm?

7. Due to the rotation of the sun about its axis, there is a relative velocity of 4 kilometers per second between the east limb as it approaches us and the west limb as it recedes. What is the magnitude of the shift, $d\lambda$ of line, $\lambda = 4283$, from its normal position, as shown in the spectrograms E and W of Plate II, § 472.

CHAPTER XXXIV

INTERFERENCE

457. Interference.—One of the most important phenomena of the wave propagation of energy, as studied in the section on Sound, was that of interference. It was seen that, under proper conditions, two or more wave trains in the same medium would combine to form regions of maximum and of minimum disturbance. In sound two similar sources, such as two tuning forks, produced beats if their frequencies were slightly different. These beats, consisting of an alternating rise and fall in the intensities of the sound, showed that sounds may be so combined as to produce not only louder sounds, but also silences. In considering the nature of the propagation of light, the question arises as to whether similar effects are to be found. Also, do light waves bend around corners as sound waves certainly do? As a matter of fact, there is a large class of important phenomena which have for their explanation the bending of light waves around corners and the mutual interference of trains of light waves. These phenomena have as yet received no other explanation, and they are accordingly strong evidences of the wave theory.

458. Colors of Thin Films.—The beautiful play of color seen in soap bubbles or in the thin films of oil on wet surfaces has its explanation in an interference effect. Occasionally cracks in ice show similar colors. The value of pearls lies in the colors produced by these effects occurring between the thin surface layers.

AB in Fig. 297 represents a vertical soap film stretched between wire supports, which has become wedge-shaped due to the liquid between the faces flowing downward. An incident beam is reflected partly at the first surface and partly at the second surface. That portion of the beam incident at C is shown leaving the film in two parts, one part due to reflection at the front surface, and one at the second. This latter beam has traveled an extra distance due to its two passages through the film. According to the corpuscular theory, an eye, receiving these two beams, would be more affected than it would be by either alone. According to the wave theory, this is not necessarily so. Thus, if the light used has a wave length λ, and the retardation caused by the double passage through the film at C is $\frac{1}{2}\lambda$, then the two beams leaving the film at C will differ in phase by $\frac{1}{2}T$, which will cause destructive interference; and if the two beams have equal intensities, the result will be that no light energy is propagated in this direction. Looking in the direction FC, the film will appear dark.

Similarly, if the retardation at C' is equivalent to one whole wave length, the two beams $C'F'$ should be in phase; and on the supposition that they are of equal intensity, the combination of the two should be four times as intense as either by itself. This is because the intensity is proportional to the square of the amplitude. The film seen in the direction of $F'C'$ appears bright. If the relative retardation at C'' is $\dfrac{3}{2}\ \lambda$, then looking along $F''C''$, the film would be black. Thus alternating bright and dark bands will cross the surface of the film.

On the basis of the discussion so far, reflection which occurs near the vertex of the wedge at A should suffer relatively no retardation, since the two surfaces are practically in contact; and this portion of the film should be bright. Experiment shows that when a film has been allowed to stand long enough to make the top of the wedge of minimum thickness, it appears black by reflected light. This means that the two reflected beams must leave the surface in opposite phase. Since this cannot be caused by difference in path, it is explained upon the basis of reflection under the two conditions. The reflection at the first surface is in air against liquid, while that at the second surface is in liquid against air. In Sound and in Mechanics it was found that reflections against rarer and against denser media introduced a difference in phase equivalent to a retardation of $\frac{1}{2}\lambda$. Assuming that this will be true also in light, there is to be added to all phase differences due to path differences, a difference equivalent to $\frac{1}{2}\lambda$, due to reflection under the different conditions existing at the two surfaces. The result will be that at *C* the total phase difference is $\frac{1}{2}\lambda + \frac{1}{2}\lambda$, or λ; at *C′*, $\frac{1}{2}\lambda + \lambda$, or $\frac{3}{2}\lambda$; at *C″*, $\frac{1}{2}\lambda + 1\frac{1}{2}\lambda$, or 2λ. *C* is therefore a bright region, *C′* dark, and *C″* bright, etc.

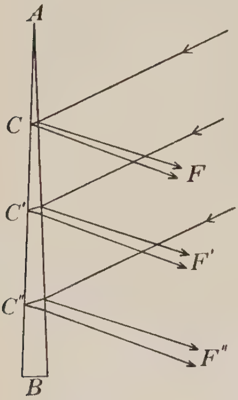


Fig. 297

If our light of a single wave length is red light, it is found that the first bright band at *C* is farther down from the vertex *A* than would be the case if blue light were used, assuming the film thickness at *A* to be zero. This shows that red light has a longer wave length than blue, since it requires a greater thickness. By this method the wave lengths of the colors of the spectrum may be shown to diminish as one proceeds from red to violet. The position of the first violet band shows that the violet wave length is about one-half the red wave length; or we may say the limits of the visible spectrum are about one octave apart. This is a much more limited range of frequencies than that which the ear is able to perceive in sound.

When a source containing all colors, for example, sunlight, illuminates the film, then, since each color has a definite wave length, any given place in the film will show a color effect due to combining all the separate effects of the individual colors. Thus, some colors will be wanting completely or partially since the film thickness there happens to be such as to cause more or less destructive interference, while those colors which leave the film in phase will be very bright, and their combination will produce the film color at this place. When the film thickness is constantly changing, as in a soap bubble, the colors which leave in phase at any place are constantly changing and therefore their resultant combination color also changes.

459. Measurement of Wave Lengths by Thin Films.—When the thin film is an air film whose thickness may be varied by measurable amounts, the change in thickness may be used to measure the wave lengths of the light used. Thus, in Fig. 298, a thin wedge of air is between the flat glass plates *A* and *C*, *C* being made of black glass. A beam of monochromatic light, reflected at normal incidence from the two surfaces of this air film, will show alternate bright and dark bands of equal width, indicated in the figure by the letters, *B*, *D*, *B′*, *D′*, *B″*.

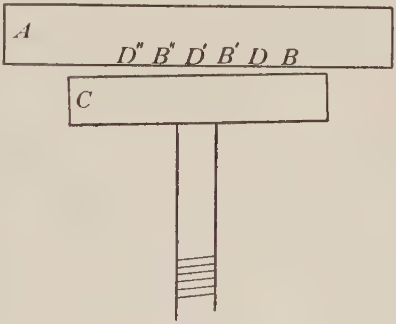


Fig. 298

The film under *B* is of the right thickness to cause a relative retardation between the two reflected beams of a whole number of wave lengths, so that these beams are in phase and their effects are added. The thickness under *D* is such as to produce between the reflected beams a difference in phase of $\frac{1}{2}\lambda$ more than at *B*. Thus there is destructive interference. At *B′* the thickness is such that the reflected beams have a phase difference of λ greater than those at *B*, and are accordingly in phase. Since the film is traversed twice to make this change of λ, the film at *B′* is $\frac{1}{2}\lambda$ thicker than at *B*. If the thickness of the films could be accurately measured at these points, their differences would equal $\frac{1}{2}\lambda$ of the light used.

If the lower plate *C* can be raised or lowered by an accurate screw movement, then, as it is slowly raised, the bright band *B* which results from a definite thickness of film will move so as to keep pace with this particular film thickness. By the time *C* has risen enough to make the thickness under *B′* equal to that originally under *B*, the bright band *B* will have moved to the position occupied by B', and B' will have moved to B''. A change in thickness of the film equal to $\frac{1}{2}\lambda$ is indicated by a shift of one bright band to the position of an adjacent bright band. If the upward movement of C, corresponding to $\frac{1}{2}\lambda$, can be measured in cm by means of the twist of the screw, then $\frac{1}{2}\lambda$ = a certain number of cm. For red light, the value of $\frac{1}{2}\lambda$ equals .00003 cm. This is not an accurate method of measuring wave lengths. Other methods are very accurate, leading to results expressible in terms of eight significant figures. Though the wave lengths of light are very small, they are among the most accurately measurable of all physical quantities. The apparatus shown in outline in Fig. 298 may be used in the reverse sense. When a bright band of known wave length changes its position to that of its nearest neighbor, the film has changed its thickness by $\frac{1}{2}\lambda$. Since these bands may be made to be several mm apart by properly regulating the angle between the plates, 1/100 of the displacement between adjacent bands may be detected. Thus $\frac{\lambda}{200}$ may be easily measured. In terms of inches, a wave length is about 1/40,000 inch; this apparatus therefore makes possible the measurements of such small displacements as $\frac{1}{200}$ of $\frac{1}{40{,}000}$ or $\frac{1}{8{,}000{,}000}$ inch. Until recently no other methods have rivaled interference methods of measuring small displacements. Now, however, such minute changes of capacitance are measurable that smaller displacements of condenser plates can be measured than can displacements of interference plates.

460. Interference from the Two Small Apertures.—The interference effects of thin films are upon a scale which make them readily observable. The effects now to be considered require special conditions to render them evident. Huygens believed that light, being propagated by a wave motion, would bend around corners, as do water waves or sound waves. The experiment shown in Fig. 299, which was made by Young, shows that this is true also of light. S is a source of light which is focused by a lens upon a pin hole in an opaque screen A. About half a meter away is a second screen B, with two pin holes about 2 mm apart, which are illuminated by light from the single hole in screen A. The beams which diverge from the two holes in B are much more divergent than would be permitted by a simple straight line propagation through the holes. Assuming that the light does bend sufficiently around the edges of these holes to permit the two beams to light a common region on the screen C, a situation is developed which should result in interference phenomena in the region ef.

The wave disturbances from the single hole in A reach the two holes in B with a certain phase difference between them which will always be constant for any radiation starting from A. A point in the region ef, so situated that the path of the light waves from one aperture in B is any whole number of wave lengths different from that from the other aperture, will make the two reinforce each other, and make the point a bright point. If, however, the path difference from the two apertures is such that the light from them arrives in opposite phase, there is destructive interference, and the point is dark. There will thus be a succession of light and dark points between e and f. Since such effects are observable, the supposition that light bends around corners has been proved. It is also true that two beams of light may be so combined as to produce darkness.

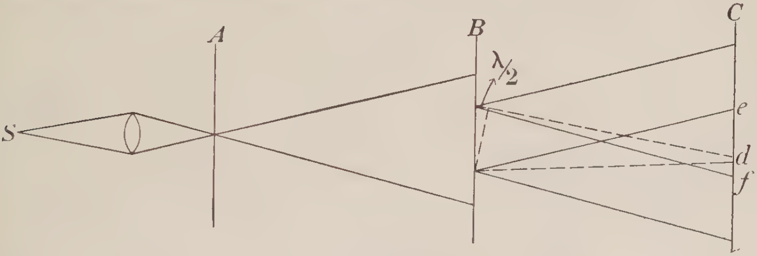


FIG. 299

461. Interference Shown by Ripples on Water.—Two trains of ripple waves from similar adjacent sources are readily formed. In Fig. 300, incident waves are shown coming from the left and striking the screen MM' in which there are two small apertures, a and c, from each of which independent wave trains diverge, in accordance with Huygens' Principle. These two trains maintain constant phase relations since they have a common source. If the circles represent at a given instant the crests of the ripples, and the troughs are represented by imaginary circles midway between the drawn circles, it will appear that there are lines along which the two wave systems are always interfering destructively, and other lines along which their effects are added. Those lines in the figure which appear to be approximately radial, but which in reality are hyperbolas, are the lines along which the two wave systems meet in the same phase; hence they produce reinforcement. In the regions midway between the radial lines, the waves from the two sources meet in opposite phase, and destructive interference results. This is apparent, since at any point in this region a crest from one wave train arrives at the same time as a trough from the other. A plane parallel to the screen MM' would show alternate regions of disturbance and quiet, the one where the two trains arrive in the same phase, and the other where they arrive in opposite phase. This is a situation quite analogous to the alternate light and dark places in the region *ef* of Fig. 299.

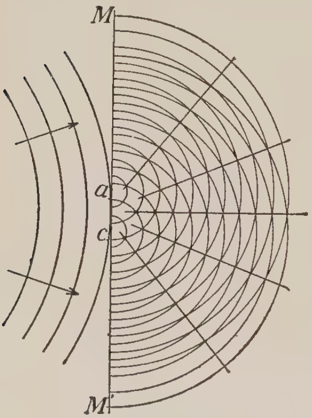


Fig. 300

462. Diffraction Through a Narrow Slit.—Those effects which result from radiation passing outside the region limited by strictly rectilinear propagation are known under the general term of diffraction. Rectilinear propagation does not permit the bending of light around the edges of opaque objects; nevertheless, such bending does occur, and it is the purpose of this topic to discuss one such case.

When a parallel beam of monochromatic light is incident upon a screen containing a slit whose width is *ab* and whose length is normal to the plane of the paper, as shown in Fig. 301, straight-line propagation would require that the screen S' be illuminated over an area of the exact size and shape of this slit in S. If the slit is wide, this is quite accurately the case. If the slit is very narrow, the resulting situation is quite different, and the light will diverge to both sides. Each point of *ab* is on a common wave front, since the incident beam is a parallel beam and normal to S. All points of *ab*, acting as sources according to Huygens' Principle, send out radiation starting in phase.

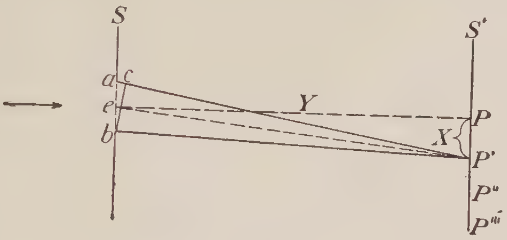


FIG. 301

Since the distances of all points between a and b from the point P at the end of a normal from the middle point of *ab* to the screen S' are practically equal, all radiation from the various point sources in *ab* reaches P in phase, and the effect is a maximum. P is accordingly a point on a bright line perpendicular to the plane of the figure.

If P' is so located that aP' is longer than bP' by one wave length λ of the light used, then $ac = \lambda$, and the rays aP' and bP' reach P' in phase. e is a point midway between a and b and the difference in the distances aP' and eP' is $\frac{1}{2}\lambda$, or these two rays would suffer destructive interference. Similarly, a point source immediately below a and one immediately below e will send rays to P' in opposite phase. Since the same relation exists for every pair of corresponding points in the two halves of the aperture, all the radiation from the aperture suffers destructive interference at P'. This point is accordingly a point in a dark line parallel to the bright line through P.

Let a point P'' be so placed that $aP'' - bP'' = \frac{3}{2}\lambda$. Then, considering the aperture to be divided into three equal parts, it is seen that any two of these parts which are next each other stand again in such a phase relation to each other that the radiation from them suffers destructive interference at P''. P'' will accordingly be illuminated by radiation from the remaining third. It is a bright point, but not so bright as P. If $aP''' - bP''' = 2\lambda$, then ab may be divided into four equal parts, and there is destructive interference between the first two and also between the second two. P''' is accordingly a point on a dark line.

These results may be generalized by expressing them by means of the following equation:

$$aP - bP = \frac{n\lambda}{2}, \text{ where } n \text{ is any integer greater than one.}$$

When n is an even number, the aperture is divided into an even number of parts and their contributions at any point, P, suffer destructive interference in pairs. All these points are dark. When n is an odd number, there is always part of the aperture which is contributing to the illumination of the point on S', and the point is bright. This brightness drops off rapidly since the effective area illuminating the screen decreases rapidly to 1/3, 1/5, 1/7, etc.

When the angles abc and PeP' are small, sin abc may be considered equal to tan PeP', and

$$\frac{ac}{ab} = \frac{x}{y},$$

where x is the distance from P to any point P' on the screen S'.

Since $aP - bP = \frac{n\lambda}{2} = ac$,

$$\frac{\frac{n\lambda}{2}}{ab} = \frac{x}{y}, \text{ or } x = \frac{n\lambda y}{2ab}.$$

It is evident from this relation that the bands are farther from P for red light than for any of the other spectral colors, since λ for red is larger than the other colors. If white light is incident upon S, the color with the smallest value of x to appear when $ac = \frac{3}{2}\lambda$ will be the violet, and red will be the last. The first bright diffraction band will then have a violet edge toward P and a red edge on its far side.

The bands will be farther from P the smaller the aperture is. This explains why these effects, due to bending around corners, are not commonly observed. It is only when openings or opaque bodies are used which are not too large in comparison with the length of light waves, that these effects are readily observable. They are always present with larger objects but generally on such a minute scale that they are not seen unless special means are used.

463. Diffraction Gratings.—A diffraction grating makes use of the fact that light will be diffracted through large angles when passing through sufficiently small openings, or reflected from sufficiently small surfaces. The type of grating to be considered here is the transmission type, in which there is a series of alternate transparent and opaque lines parallel to each other on a suitable surface. Thus, G in Fig. 302 represents such a grating in which a, c, d, etc., are the upper edges of transparent lines between opaque lines. The lines are assumed normal to the surface of the page. There are usually hundreds or thousands of these lines per inch on an actual grating, which may itself be several inches wide.

When a parallel beam of monochromatic light is incident normally upon the grating, each opening diffracts radiation in all directions. If such radiation as proceeds in the direction of the incident beam is first considered, all such radiation is focused by the lens L at its principal focus F. This is called the undeviated image. Next consider the radiation from each opening which has such a direction that the distance $aa' = \lambda$

for the light used. This means that the light from each opening is retarded by one whole wave length with respect to the light from an adjacent opening. *fa″* is a wave front proceeding in the direction making an angle *aca′* with the undeviated rays. Since *fa″* is a plane wave, the lens will bring it to a point focus in its focal plane, as at *F′*. *F′* is a bright image since the rays focused are all in phase, having started from the wave front *fa″*.

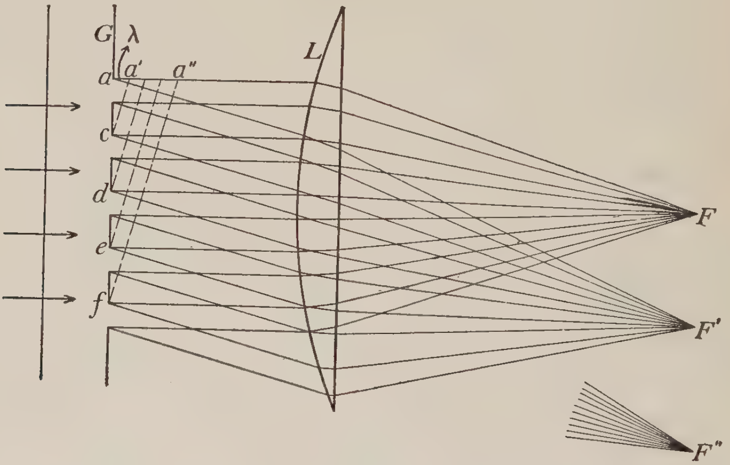


FIG. 302

Again consider the rays diffracted by the openings in such a direction that the retardation *aa′* becomes 2λ instead of λ. Such rays will make a larger angle with the undeviated rays than *aca′*, and will be in phase when focused by the lens at *F″*. Obviously other directions for which the rays would be in phase when focused are those for which *aa′* becomes 3 λ, 4 λ, 5 λ, or any whole number times λ.

The image *F* is called the undeviated image, *F′* the first order line, *F″* the second order, etc. Obviously these rays which have been deviated downward in the figure could have been deviated upward just as well, with equal retardations. There are accordingly two first order lines at equal distances on either side of *F*, two second order lines etc.

It can be shown that radiation proceeding in any other directions than these just discussed will suffer extinction through interference, the more sharply the greater the number of lines in the grating.

464. Measurement of Wave-Lengths By Means of Gratings—If a transmission grating is substituted for the prism on the spectrometer in Fig. 291, and a source of light giving a bright line spectrum is placed before the slit *s*, direct measurements of wave lengths are possible. The spectrometer is provided with an accurately graduated circle by which the position of the telescope may be exactly determined. The lens *L* of Fig. 302 is replaced by the lens *T* of the telescope, while the images of Fig. 302 become colored images of the slit *S*, and lie in the focal plane of the lens *T*. They are viewed by the eyepiece *E*, which permits one to bring the lines into coincidence with a vertical cross hair placed in the focal plane of the lens *T*, and thus to locate them accurately. When the telescope is moved from the position of the direct image to that of the first order line, it has turned through an angle equal to the angle aca' of Fig. 302. The wave length λ to be measured is related to this angle and the grating constant by the following equation, $\lambda = ac \sin aca'$. The distance ac between adjacent lines of the grating is called the grating constant. It can be accurately measured by a dividing engine and can be regarded as a known quantity. The angle aca' is determined by direct measurement as described. When the telescope is focused upon the second order line, the equation becomes

$$2\lambda = ac \sin \Theta_2$$

where Θ_2 is the observed angle of deviation of the second order line. In general, the equation for the determination of wave lengths is

$$n\lambda = ac \sin \Theta_n,$$

where n refers to the order of the spectrum used for the measurement. The same λ can thus be measured in its several orders.

Both Fig. 302 and the equations show that the deviations increase with the wave lengths. With the slit illuminated by a source of complete radiation, the violet, having the shortest wave length, is deviated least, and the other colors through larger angles, as given by the equation. The continuous spectra which result on either side of F will have their violet ends next to F. The second orders of these continuous spectra will have 2 λ for the violet, not much beyond 1 λ for the red of the first order. The second and third order spectra overlap. As the angle of deviation becomes large, there may be several orders overlapping at any one place. This sometimes causes trouble in interpreting results. This effect is not observed with a prism instrument, since there is only one spectrum produced. Both the prism and grating spectrometers have their particular advantages and disadvantages.

By making the grating constant ac very small, as when there are from 10,000 to 20,000 lines per inch, large deviations and extended spectra result. There are other types of gratings and other ways of mounting them which permit great dispersion. Thus Rowland, who first successfully made reflection gratings by ruling optically plane metal surfaces with a diamond point, was able to make a photographic spectrum of the light from the sun which was several feet from end to end, approximately six feet for the visible spectrum from one grating having 20,000 lines per inch.

PROBLEMS

1. Two rectangular pieces of plane glass are laid one upon the other on a table. A thin strip of paper is placed between them along one edge so that they are not quite parallel. They are illuminated by a beam of sodium light falling vertically upon them. For each centimeter measured perpendicular to the edge where the paper was inserted, ten bright and ten dark interference bands may be counted. Find the angle between the two plane surfaces.

2. With the transmission grating having 500 lines per mm, through what angle will radiation, incident normally, be deviated from the direction of the undeviated image, when the wave length is 6000 A? Give the results for the first, second and third order spectra.

CHAPTER XXXV

POLARIZED LIGHT

465. Polarization by Reflection.—If it is assumed from the phenomena of interference that light is propagated by a wave motion, the question arises whether this wave propagation is produced by a system of transverse waves or longitudinal waves. There are some phenomena of light which have their explanation in the theory that light waves are transverse waves. One of the first observations which led to this belief was that light reflected from glass sometimes has very different properties after reflection. Investigation showed that for a particular angle of incidence, about 57°, the effect is most marked. The observed effect is that light once reflected is not often again reflected in the usual way. Experiment showed the conditions under which this reflected beam would again reflect in the normal way, and also the conditions when there would be no second reflection at all. These two cases are shown in Fig. 303. At the left an incident beam is reflected at *A* and suffers a second reflection at *B*. The mirrors are of unsilvered glass, or, better, of black glass. The angles of incidence are 57° in each case. When the mirror *B* is rotated through an angle of 90° upon the beam *AB* as an axis, as shown in the figure at the right, there is no reflection. This is a remarkable fact. Before considering its possible explanation, the case of transmission will be considered.

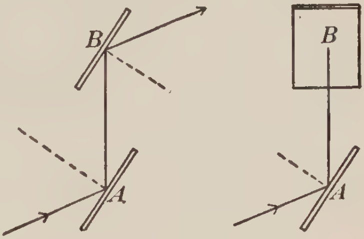


FIG. 303

466. Polarization by Refraction.—When a beam of light passes perpendicularly through a plate of tourmaline crystal cut with the surfaces parallel to the optic axis of the crystal, the emerging beam appears to have a property similar to that of the reflected beam which was refused a second reflection. A second tourmaline plate parallel to the first will refuse transmission when at right angles to the position in which it most completely transmits, the crystal having been rotated upon the beam as an axis.

In both cases just considered, a beam of common light has been changed into one having the unusual property of selection upon subsequent reflection or refraction. A wave form traveling along a rope may have its plane of vibration vertical, as in Fig. 304. In this case the rope will continue to hand on the wave motion, even though it is passed through a vertical slit such as *A*, if *A* alone is used. If slit *B*, 90° from *A*, is used, then the wave as shown will be refused transmission. Any wave, however oriented, will be completely suppressed when both *A* and *B* are used and are at right angles to each other. *B* will transmit only the horizontal components of the incident beam, and *A* will refuse transmission to all such disturbances.

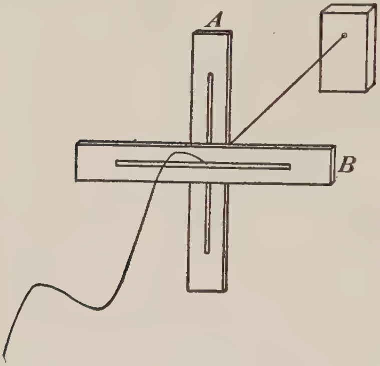


Fig. 304

Under no circumstances would a longitudinal wave be thus refused passage. It is only by considering that our light beam, which under certain conditions is refused reflection or refraction, has properties similar to those of the wave on the rope that we are able to explain these phenomena.

467. Explanation of the Preceding Phenomena.—It is assumed that the light wave is transverse in type rather than longitudinal, and that reflection at the surface *A* suppresses all the components of the vibrations of the incident beam normal to the line of propagation, except those parallel to the reflecting surface. The reflected beam will accordingly have all its vibrations parallel to the surface of the mirror A and normal to the beam AB. It is said to be plane polarized. Thus, with the mirror B in its first position, reflection would again suppress all vibrations not parallel to the mirror surface. Since B is parallel to A, and the beam AB contains only vibrations parallel to surface A, there are no components to be suppressed by B, which accordingly reflects the beam completely. When B is rotated upon AB as an axis, the components of the incident vibrations normal to the surface of B become progressively larger, and when B has rotated 90° the incident vibrations have no components whatever parallel to the surface of B. There is accordingly no reflection. The parts of the incident beams which are not reflected, due to the fact that their vibrations are normal to the reflecting surface, are refracted. If the glass is black glass, these refracted parts are quickly absorbed.

Some crystals are doubly refracting when a beam of ordinary light is sent through them in certain directions. The two beams after emerging, if examined by a mirror, will each be found to be plane polarized, since for each beam there is one position of the reflecting mirror at which no reflection occurs. These positions of the mirror for the two beams are at right angles to each other. It is concluded, therefore, that the two refracted beams are polarized at right angles to each other. In the case of tourmaline, the crystal completely absorbs one of these two, so that only one emerges. When this is incident upon a second plate of tourmaline, the latter may be rotated about the incident beam as an axis, and a position can be found so that the beam is transmitted. When the crystal is rotated 90° from this position, the incident vibrations are in the plane for which the crystal absorbs the beam, and no light is transmitted.

468. Nicol Prisms.—Nicol prisms are used to obtain plane polarized beams of considerable intensity, and their use is based on the fact that iceland spar, of which they are made, is doubly refracting. The prisms are so constructed as to suppress one of these beams. Thus, in Fig. 305 a beam of ordinary light incident upon the surface A is doubly refracted. These two refracted beams are incident upon the surface CB, which is a layer of Canada balsam used to cement the two parts of the prism together. Since there is double refraction there are two velocities of propagation in the crystal. The balsam is not crystalline, and both rays have the same velocity in it. This common velocity happens to be greater than one of the velocities in the crystal and smaller than that of the other. When a ray seeks to pass from one medium into a second where it has a greater velocity, there will be total reflection if the angle of incidence in the denser medium is greater than the critical angle for the two substances. These prisms are so made that the angle of incidence exceeds the critical angle for the beam whose velocity is greater in the balsam than in the crystal. This beam is accordingly totally reflected. The other beam passes through since its angle of incidence as it goes from the balsam to the crystal is less than the critical angle. Thus the emergent beam is plane polarized. As these prisms can be made quite large, very strong beams of polarized light are thus obtained.

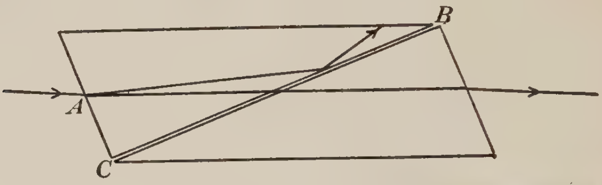


FIG. 305

Just as a second tourmaline crystal refuses transmission to the beam passing through the first when properly oriented, so a second Nicol in line with the first will refuse to transmit if the beam received from the first has its vibrations in the plane which the second Nicol does not pass. At other relative positions light is passed. The maximum is transmitted when the position is 90° from that of non-transmission. Since one of these prisms appears perfectly transparent when one looks at any bright object through it, it is very striking to find that under the proper conditions no light at all can pass through.

CHAPTER XXXVI

SPECTROSCOPY

469. Spectral Limits.—So far only visible spectra have been considered. Early in the study of radiation, however, it was discovered that luminous sources give out radiations to which the eye is not sensitive. These radiations are found to form extensions to the visible spectrum at each of its ends. Radiations whose wave lengths are shorter than the violet are called ultra-violet. Recent research has extended this region several octaves beyond the shortest visible radiation. Radiation beyond the red end of the visible spectrum is called infra-red and extends over many octaves. The invisible spectrum is of much greater extent than the visible. The unit in which wave lengths are expressed is the Angstrom, which is $\frac{1}{10,000}$ μ, where μ is $\frac{1}{1000}$ mm. The yellow light of the sodium flame has a wave length of about 5890 A. The entire range of the visible spectrum is from 4000 A. to 7500 A. The shortest ultra-violet determinations record wave lengths of less than 100 A.; the longest infra-red, of 3,000,000 A. The ultra-violet region is recorded photographically, the extreme portion of it *in vacuo*. The infra-red observations are recorded by means of the heat effect of the radiation. Here bolometers, thermopiles and radiometers find use.

Of shorter wave length than the extreme ultra-violet are the X-ray waves, and still shorter are the γ radiations given off by certain radioactive substances. The range of these radiations is from a few tenths of an Angstrom up to the ultra-violet region. The shortest of all are the cosmic rays which are found in interstellar space. Their estimated wave length is of the order of .0001 A.

Longer than the longest infra-red radiation are the electromagnetic waves used in radio. The gap between this radiation and the infra-red has been closed. Radio wave lengths from .3 mm to many km are measurable.

All these radiations are of the same kind and may be called electromagnetic. Their only difference is one of frequency. The cosmic rays have the greatest frequencies and the shortest wave lengths; the radio waves have the lowest frequencies and the longest wave lengths. They all form a single spectrum.

470. Kinds of Spectra.—Continuous spectra and bright line spectra have already been described in §447. There is a third kind, called absorption spectra. When complete radiation—that is, radiation which when resolved into its spectral components gives a continuous spectrum—is passed through a gas and the emerging radiation is examined spectroscopically by means of a prism or grating spectrometer, certain regions or narrow lines may be missing from the continuous spectrum. This radiation which has been absorbed by the gas constitutes its absorption spectrum. In appearance it consists of dark regions more or less wide, or dark lines upon a background of a continuous spectrum.

Line 4 in Plate I of §447 shows a dark line spectrum. It is the sun's spectrum which, as emitted by the hot interior of the sun, is continuous, but which has certain wave lengths removed through absorption by the cooler gases of the outer atmosphere of the sun or by the earth's atmosphere. Lines A and B are due to oxygen; the two at D to sodium; the one at F to hydrogen, and those at G, H, and K to calcium. The scales are in Angstrom units, and it is possible to estimate the wave lengths of the lines with some degree of accuracy by means of them.

I. Continuous spectra originate in luminous solids, liquids and gases which are sufficiently dense. There are portions of spectra which appear continuous which are obtained from rarefied gases.

II. Bright line spectra originate in gases rendered luminous by any one of several possible ways—high temperature, electrical discharge, etc. Solids may be vaporized and the vapors rendered emitting in the electric arc. The material is introduced into the arc by placing it in holes bored into the ends of the carbons. Or if the materials are conducting, they may be made the terminals between which discharges from electrical condensers occur. Small amounts of the terminals vaporize and the vapor is rendered highly emitting by the discharge. Such bright line spectra are usually called spark spectra, in contrast with the arc spectra.

III. The origin of absorption spectra has been described earlier in this article.

471. Information to be Obtained from Spectra.—Only a few of the many kinds of information to be derived from a study of spectra can be considered here. It was very soon discovered that line spectra, both emission and absorption, were characteristic of the elements. So true is it that each element has its own spectrum that a new spectrum means a new element, or possibly, in these days, an element somewhat made over by the physicist, the extent of the change being shown by the change in spectrum.

The spectroscopic analysis of materials which can be vaporized or used in absorption cells is a common practice. The sensitivity of the method is, however, too great for most analyses, since the most minute quantities of materials are detectable by it, and few chemical preparations are sufficiently pure to pass such a test. The failure to find the lines of a substance is conclusive evidence of its absence. Lately methods of quantitative analysis have been developed by which the per cent of impurities can be estimated by comparison of the spectrum of an unknown composition with spectra of known compositions.

By means of spectra we obtain information not only about materials upon the earth, but from solar, stellar, and nebular spectra we learn of the materials of which these bodies are composed. The presence of the lines of any element in the spectrum of a star is conclusive evidence that that star contains the element. Until recently, the failure of the spectrum of an element to appear indicated its absence. Recent studies of spectra obtained from a single element, when it has been made progressively more simple by taking away its outer electrons one by one, show that with the removal of each electron a new type of spectrum is obtained. This spectrum in general is displaced toward the short wave lengths. Where the extremely high temperature of certain stars has resulted in atoms losing their outer electrons more or less completely, their spectra may have moved so far into the ultra-violet region as to be no longer transmitted through the earth's atmosphere. Absence of spectra thus does not indicate absence of the element but rather the temperature of the star. This knowledge is making it possible for astrophysicists to develop the life history of stars.

When the source of radiation is placed in a strong magnetic field, the spectral lines separate into several lines. The number of the components and their separation is characteristic of the line and the strength of the field. The separation of certain lines, when the spectra of sun spots are studied, has shown that these spots possess a magnetic field. This knowledge is of great importance in solar theory or in any theory connecting the sun with earth phenomena.

The list of illustrations of the information to be obtained from a study of spectra is very incomplete, but it may serve to show its possibilities and to explain why there is such great interest in the subject at the present time. Most important of all for the physicist and the chemist is the information which the study of spectra gives about atomic and molecular structure. Spectra undoubtedly furnish the most extensive and accurate data concerning atoms and molecules.

472. Doppler Effect.—In addition to the kinds of information just enumerated, there is the knowledge of velocities to be gained through the Doppler Effect. If light is propagated by a wave disturbance there should be, as in sound, changes in wave length when the source advances or recedes in the line of sight with sufficient velocity. Such velocities are found in stars and nebulæ.

In Fig. 306 the source of the waves is moving to the left with a velocity comparable to that of the wave motion. As a result there is a very evident shortening of the waves on the left and a corresponding lengthening on the right. The change in wave length is given by the expression

$$d\lambda = \lambda \frac{v}{V},$$

where v = the velocity of the source with respect to the observer, and V that of the light, while λ is the wave length of the light of a source at rest and $d\lambda$ the change in wave length due to the motion of the source.

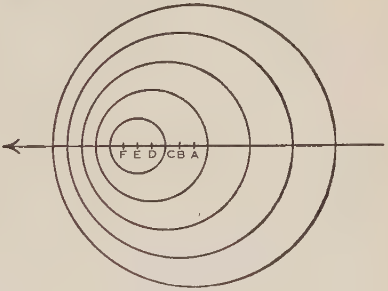


Fig. 306

When a stellar spectrum shows lines which are characteristic of an element, but all the lines are of slightly smaller wave lengths than are correct, it is assumed that the star is moving rapidly toward the observer. By measuring the change in wave length of the displaced lines, it is possible to compute the velocity of approach in the lines of sight. The component at right angles cannot be measured by this method. Stellar velocities may be as large as one hundred kilometers per second.

When a star shows a spectrum with wave lengths periodically shortened and lengthened, the approach and recession of the star necessary to explain the shift of the lines is accounted for by assuming that the star is one of a pair of double stars rotating about their common center of gravity. By measuring the shifts in these cases, it is possible to compute the shape of the orbit of the star, its period of revolution and its velocity in the orbit.

473. Regularities in Spectra.—Some elements give comparatively simple spectra containing only a few lines; others give very complicated spectra containing thousands of lines. Regularities, first found under favorable conditions, have been rapidly multiplied, until many elements have their wave lengths quite completely arranged in series. Such a regularity or series is seen in Fig. 307. It will appear that the lines of a series begin with strong lines and, as the wave lengths decrease or the frequencies increase, the lines become progressively closer together and at the same time weaker. There is a definite limit at the short wave length end of the series beyond which its lines do not extend. This limit is indicated by the dotted line *L*. Experiment seems to show that as the limit is approached the lines are packed closer and closer together so that it becomes impossible to separate them by present methods. As many as forty lines have been measured in some series. There are elements which have groups of two, three, or more lines instead of the single lines of Fig. 307. These series are known as doublet, triplet and multiple series respectively.

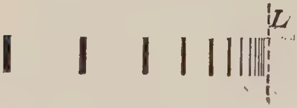


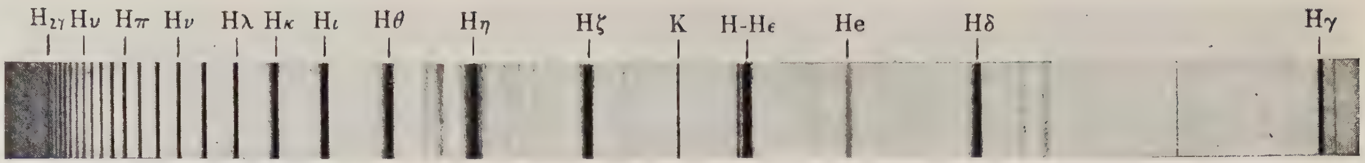
FIG. 307

474. Series of Atomic Hydrogen.—The series of hydrogen are among the simplest known and are the only ones which will be discussed here. The first of these series to be discovered was the one lying partly in the visible portion of the spectrum. The wave lengths, measured in Angstroms, are given in the first column of the accompanying table.

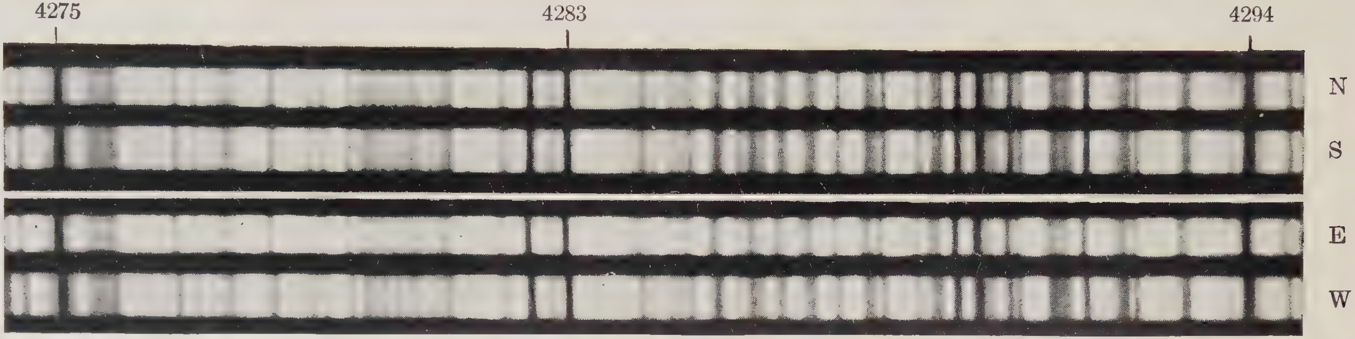
Balmer, after whom this series is named, found that the measured wave lengths could be accurately represented by the following formula,

$$\lambda = c \cdot \frac{m^2}{m^2 - 4},$$

where *c* equals 3646 A. and *m* takes on in succession the values 3, 4, 5, 6, 7, etc. The values of the wave lengths computed by this formula are given in the second column of this table,



SPECTRUM OF THE STAR ξ TAURI, SHOWING THE BALMER SERIES OF HYDROGEN LINES
Photographed by R. H. Curtiss, Detroit Observatory



DOPPLER EFFECT DUE TO ROTATION OF THE SUN
Lines N and S show spectra of points near the poles with no shifts. East limb of sun approaches and west limb recedes from us, with shifts to violet and red respectively
Photographed by Adams

PLATE II

(From Duncan's *Astronomy*, published by Harper & Brothers)

while the last column shows the differences between the measured and computed values. The formula represents the observed data with remarkable accuracy. It is also noticeable that the successive values are obtained by using as the value of m the successive integers beginning with $m = 3$. Plate II shows an excellent photograph of this series taken by R. H. Curtiss of the Detroit Observatory, and shows the spectrum of the star Zeta Tauri, the hydrogen lines being indicated by the H's with the Greek letter subscripts. In addition, He is a helium line and H and K are calcium lines.

Experience has shown that with series the frequencies of the lines are more convenient quantities to work with than their wave lengths. The relation connecting frequency f and wave length λ, is $\lambda = \frac{V}{f}$, where V is the velocity of light. By its aid the Balmer formula may be expressed in terms of frequency instead of wave length; thus:

$$\lambda = c \cdot \frac{m^2}{m^2 - 4} = \frac{V}{f};$$

$$f = \frac{4V}{c} \cdot \frac{m^2 - 4}{4m^2}, \text{ or}$$

$$f = \frac{N}{4} - \frac{N}{m^2},$$

where the new constant N is $\frac{4V}{c}$.

As before, m takes in turn the values of the successive integers beginning with 3. This expression shows f to be the difference between two quantities; the first, $\frac{N}{4}$, is a constant, and the second, $\frac{N}{m^2}$, varies with the successive values of m. When m becomes very large this second quality becomes small and may be neglected, and

$$f = \frac{N}{4}.$$

This particular value of f may be considered to be the frequency of the limit which is approached more closely by the lines of the series, the larger m becomes. Series are described by telling how far each line is from the series limit. $\frac{N}{m^2}$ gives these distances, $\frac{N}{3^2}$ for the first line, $\frac{N}{4^2}$ for the second line, and so on.

THE BALMER SERIES

Observed	*Calculated*	*Difference*
6563.07	....	......
4861.57	4861.52	+ 0.05
4340.53	4340.63	− 0.10
4102.00	4101.90	+ 0.10
3970.33	3970.22	+ 0.11
3889.15	3889.20	− 0.05
3835.51	3835.53	+ 0.02
3798.00	3798.04	− 0.04
3770.73	3770.77	− 0.04
3750.27	3750.30	− 0.03
3734.53	3734.51	+ 0.02
3721.98	3722.08	− 0.10
3712.13	3712.11	+ 0.02
3704.01	3704.00	+ 0.01
3697.28	3697.29	− 0.01
3691.70	3691.70	± 0.00
3686.96	3686.97	− 0.01
3682.94	3682.95	− 0.01
3679.52	3679.49	+ 0.03
3676.51	3676.50	+ 0.01
3673.87	3673.90	− 0.03
3671.53	3671.48	+ 0.05
3669.55	3669.60	− 0.05
3667.83	3667.82	+ 0.01
3666.25	3666.24	+ 0.01
3664.74	3664.82	− 0.08
3663.55	3663.54	+ 0.01
3662.36	3662.40	− 0.04
3661.31	3661.35	− 0.04
Theoretical limit	3646.13	......

475. Generality of Formula.—The formula, $f = \frac{N}{4} - \frac{N}{m^2}$, first developed and used for the Balmer series of hydrogen, has been found to have a wider application. Later other hydrogen series were found, one in the ultra-violet beyond the violet end of the visible spectrum, by Lyman, and another beyond the red end in the infra-red, by Paschen. The Lyman series is given by the expression,

$$f = \frac{N}{1} - \frac{N}{n'^2},$$

where n' is substituted for m and equals 2, 3, 4, 5, etc.; while

$$f = \frac{N}{9} - \frac{N}{n'^2},$$

where $n' = 4, 5, 6, 7$, etc., represents the Paschen series. It is evident that a single expression will represent all three of these series, namely,

$$f = \frac{N}{n^2} - \frac{N}{n'^2}.$$

$n = 1$ gives the Lyman series; $n = 2$ the Balmer, and $n = 3$ the Paschen, when appropriate values of n' are used. This suggests that other series, with n taking in turn the values $n = 4$, $n = 5$, etc., may exist. Pfund and Brackett have in fact found such series.

Not only has the formula in its generalized form been found to represent the various hydrogen series equally well, but in a slightly modified form it will often represent series of other elements. It has been found necessary to alter slightly the integers of the denominators in order to make the expression suitable for other elements.

As so altered, the expression becomes

$$f = \frac{N}{(n + a)^2} - \frac{N}{(n' + b)^2},$$

where a and b are constants for a single series only, being generally small fractions so that the denominators do not differ greatly from n^2 and n'^2. N for various elements differs only slightly from its value for hydrogen.

The general applicability of this formula to series of different elements, together with the fact that N is sensibly a constant for all elements, leads to the belief that this formula at first considered an empirical relation only, has really a physical basis, that is, it can be interpreted in terms of atomic structure.

476. The Hydrogen Atom.—The modern idea of the structure of the atom was given in §258. In the case of the hydrogen atom, the nucleus consists of a single proton, the unit of positive electricity. There being but the single positive charge of the nucleus, only one revolving electron is required by the normal atom. This atom is accordingly the simplest which can be made from the two materials, protons and electrons, out of which all atoms, however heavy, are built. Possibly the most characteristic property of any atom is its spectrum, and Bohr has made such assumptions about this atom as will permit it to radiate a spectrum in entire agreement with the one measured. These assumptions made in the case of hydrogen have been fruitful when applied to heavier and consequently more complex atoms, so that they appear to be justified by their success. In fact, whenever atomic or molecular systems emit or absorb radiation, similar assumptions have to be made which have no justification in terms of the large-scale phenomena with which we are familiar, if the observed data are to be accounted for theoretically. These assumptions are necessary to any satisfactory explanation of the line spectra which radiating gases emit. They reveal that the processes of nature within the atom must be described in terms which are new to large-scale mechanics and electromagnetism.

Bohr postulated that the hydrogen atom could exist in a number of states of different energy content, due to the revolution of the electron about the proton in a number of selected circular orbits. These orbits were chosen by imposing the condition that the electron could occupy only such orbits as would cause its moment of momentum multiplied by 2π to equal an integral multiple of the universal constant h, the Planck constant (see §411). Thus,

$$(1) \quad 2\pi mva = nh,$$

where m is the mass of the electron, v its linear velocity in its orbit, a the radius of the orbit, and n is any integer. When $n = 1$, the electron is revolving in the smallest orbit; when $n = 2$, in the orbit of next larger size, etc. In any one of these orbits its energy remains constant, though previous ideas would require that the electron emit radiation continuously. The possible radii of the permitted orbits, the velocities in the orbits and the energy of the atom associated with each orbit can be computed by using as an additional relation the equality which must exist for each orbit between the centrifugal force of the revolving electron and the electric attraction between electron and proton; thus

$$\frac{mv^2}{a} = \frac{e \times e}{a^2}.$$

$$(2) \quad mav^2 = e^2.$$

From

$$(1) \quad mav = \frac{nh}{2\pi};$$

or, squaring

$$(3) \quad m^2a^2v^2 = \frac{n^2h^2}{4\pi^2}.$$

Dividing (3) by (2),

$$ma = \frac{n^2h^2}{4\pi^2e^2}.$$

$$(4) \qquad a = \frac{n^2h^2}{4\pi^2me^2}.$$

The radius varies according to n^2, where n takes in turn the values 1, 2, 3, 4, etc. The second orbit has 4 times the radius of the first, the third orbit 9 times, etc. The relative sizes of the first six orbits are shown in Fig. 308. The smallest orbit has a radius of r, and the others in order have radii of $4r$, $9r$, $16r$, $25r$, and $36r$. Inserting the numerical values of the quantities in the expression for a, it is found that the result is 5×10^{-9} cms for n equals 1, a value of the right size.

The velocity of the electron in any one of these orbits is found by inserting in (3) the value of a^2 from (4); thus

$$(5) \quad v = \frac{2\pi e^2}{nh},$$

or the velocity in the successive orbits varies inversely as the successive integers, 1, 2, 3, 4, etc. The velocity in each orbit is shown in Fig. 308 in terms of the velocity in the first orbit.

The energy of the atom is partly kinetic due to the revolving electron, and partly potential due to the attraction between the positive proton and negative electron. The kinetic energy is

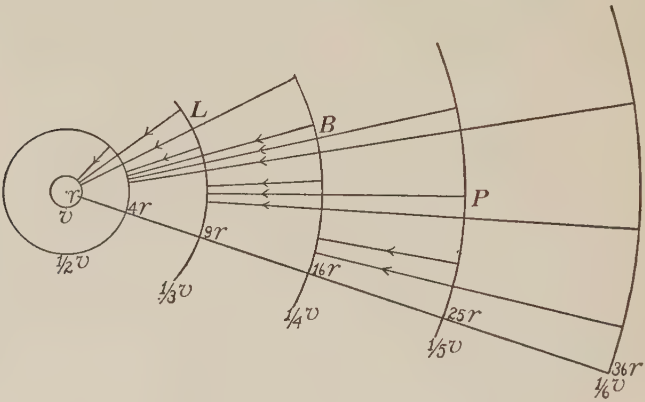


FIG. 308

$$\text{K. E.} = \frac{1}{2} mv^2.$$

Substituting v^2 from (5),

$$= \frac{1}{2} m \cdot \frac{4\pi^2 e^4}{n^2 h^2}, \text{ or}$$

$$(6) \quad = \frac{2\pi^2 m e^4}{n^2 h^2}.$$

The potential energy is

$$\text{P. E.} = -\frac{e \times e}{a} = -\frac{e^2}{a}.$$

A negative sign is used because work is done against the electric attraction between nucleus and electron when the orbit is enlarged. This means that the potential energy increases when a increases. A fraction can increase when its denominator increases only when it is negative. Using the value of e^2 from (2),

$$\text{P. E.} = -\frac{mav^2}{a},$$

$$= -mv^2;$$

or, substituting again v^2 from (5),

$$= -\frac{4\pi^2 me^4}{n^2h^2}. \tag{7}$$

The total energy is

$$\text{E} = \text{K.E.} + \text{P.E.}$$

$$= \frac{2\pi^2 me^4}{n^2h^2} - \frac{4\pi^2 me^4}{n^2h^2}$$

$$= -\frac{2\pi^2 me^4}{n^2h^2}. \tag{8}$$

If it were not for the negative sign, this would indicate that the atomic energies due to the different orbits varied inversely as the square of the successive integers, 1, 2, 3, 4, etc., the second state having one-fourth the energy of the first, etc.; however, the decrease of this negative quantity with increasing values of n means increase in the energy of the atom.

477. Energy Levels.—When n is given its successive values beginning with 1, E takes the following values:

$$-\frac{2\pi^2 e^4 m}{1^2h^2}, \text{ or } -\frac{2\pi^2 e^4 m}{h^2} = -\text{E}_1$$

$$-\frac{2\pi^2 e^4 m}{2^2h^2}, \text{ or } -\frac{2\pi^2 e^4 m}{4h^2} = -\text{E}_2 = -\frac{1}{4}E_1$$

$$-\frac{2\pi^2 e^4 m}{3^2h^2}, \text{ or } \frac{2\pi^2 e^4 m}{9h^2} = -\text{E}_3 = -\frac{1}{9}E_1, \text{ etc.}$$

To represent these results graphically, the values $-E_1$, $-E_2$, etc., are plotted below a base line since they have a negative sign. Fig. 309 shows the hydrogen energy level diagram.

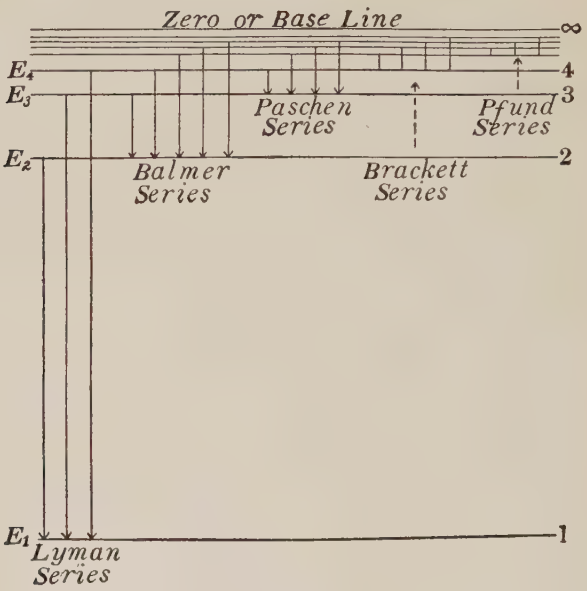


Fig. 309

If line 1 is at the correct distance to represent $-E_1$, or the energy of the atom when its electron is revolving in its first or smallest orbit, then line 2, one-fourth the distance of 1 below the zero line represents the atomic energy when the electron is in the second orbit, etc. It is evident that the differences in energy between adjacent levels diminish rapidly as n increases, or, in terms of the atomic structure, the changes in atomic energy as the electron passes from one possible orbit to the next become progressively smaller as the outer orbits are occupied.

478. Derivation of the Formula for the Hydrogen Series.—In addition to assuming that the revolving electron may occupy only certain orbits, Bohr had to assume also that when the atom changes its state upon the passage of its electron from one orbit to another, the accompanying change in atomic energy is accounted for by radiation; that this radiation is of a single frequency f, and is proportional to the energy change. These assumptions are contained in the equation

$$hf = E_{\mathrm{n}} - \mathrm{E}_{\mathrm{n}'},$$

when E_n = the atomic energy in the final state and $E_{\mathrm{n}'}$, the energy in the initial state; h, the proportionality factor, is again the Planck constant. Should the electron shift from the second to the first orbit, $\mathrm{n}' = 2$ and $n = 1$, and

$$hf = \frac{2\pi^2e^4m}{1^2h^2} - \frac{2\pi^2e^4m}{2^2h^2},$$
$$= \frac{2\pi^2e^4m}{h^2}\left(\frac{1}{1^2} - \frac{1}{2^2}\right).$$

In general,

$$hf = \frac{2\pi^2e^4m}{h^2}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right), \text{ and}$$
$$f = \frac{2\pi^2e^4m}{h^3}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right).$$

All the quantities in the fraction $\frac{2\pi^2e^4m}{h^3}$ are known. It is found, when their values are inserted, that the result is equal to the value of N in the empirical formula for the hydrogen series,

$$f = \mathrm{N}\left(\frac{1}{n^2} - \frac{1}{n'^2}\right).$$

It has thus been possible to derive the expression for the hydrogen series if the Bohr assumptions are used. The principal value of these assumptions lies in the fact that they find general application in developing the theories of the structure and spectra of the heavier and more complicated atoms.

When atomic hydrogen is made emitting, as by passing an electric discharge through it, some of the atoms remain in their normal states with their electrons in the first orbit, while others are found in various states of excitation due to their electrons being in different ones of the permitted orbits. The time an atom remains in an excited state is usually very small, about 10^{-8} seconds, so that at any instant some of the atoms in the first, second, third, etc., stages of excitation are returning to the normal state. These transitions, each of which causes the emission of a spectral line, are shown as electron transitions from orbits 2, 3, 4, etc., to orbit 1 of Fig. 308, or as the equivalent energy changes in the energy diagram, Fig. 309, where the transitions are from the higher levels, 2, 3, 4, etc., to the lowest level, 1. Such transitions, which leave the atom in its normal state, result in the emission of the lines of the Lyman series, *L* of Fig. 308.

Some of the atoms in the second stage of excitation, *i.e.*, with their electrons in the third orbit, will have their electrons fall to the second orbit; this transition results in the emission of the first line of the Balmer series, *B* of Fig. 308. The other lines of this series result from transitions from the fourth, fifth, etc., orbits to the second orbit. The same transitions are also shown on the energy level diagram.

Transitions from the outer orbits to the third orbit give rise to the Paschen series, *P* of Fig. 308. The Brackett and Pfund series are accounted for in a similar way.

479. Heavier Atoms and Their Radiation.—The heavier atoms are, the greater the resultant nuclear charge and the consequent number of outer electrons. Transition of electrons between the inner orbits of the heavy atoms results in large changes of atomic energy which, according to the Bohr assumption,

$$hf = E_n - E_{n'},$$

will give radiation of great frequency or very short wave length, *i.e.*, X-rays. Transition of the outermost electrons results in a wide range of energy changes and produces spectra of the kind discussed in the case of hydrogen, containing, however, many more lines and of much more complicated series. The type of series, doublets, triplets, quartets or higher multiplicities, depends upon the number of outer electrons taking part in the radiation. The distribution of these spectra ranges from the extreme ultra-violet, where the wave lengths are no longer than those of the X-ray radiation, through the visible into the infra-red, where the longest waves are comparable with those given out by moderately hot bodies.

480. Methods of Excitation.—In their normal states atoms have that distribution of electrons which results in a minimum of atomic energy. This means in general that the smallest of the permitted orbits is occupied. When, by any means, one or more of the electrons of an atom is made to occupy larger orbits, the atom takes on energy and is said to have a degree of excitation determined by the occupied orbits. When the energy absorbed by the atom is sufficient to remove one of its electrons completely from the atom, the atom has become singly ionized. If two are removed, the atom is doubly ionized, etc.

There are several ways by which atoms may be excited and ionized. They may absorb radiant energy, as discussed in §411 on the photo-electric effect. They may be bombarded by streams of electrons, as described in §364. This method has many advantages, one of which is that the velocities of the bombarding electrons are under control and the energy available for transfer to the atom by impact may be any desired amount. It is a remarkable fact that only certain amounts of energy may be given by bombarding electrons to the atoms they strike. This is interpreted to mean that the atoms have certain definite energy loads.

A third method of excitation arises when the temperature is so elevated that atomic collisions become sufficiently violent to disturb the normal distribution of the electrons. In these cases some of the electrons may move to outer orbits or be completely removed from the atom. Other important ways of exciting atoms are used, but need not be discussed here.

481. Molecular Spectra.—Atomic spectra have been accounted for by assuming that atoms exist in a number of states due to their electron configurations, and that when electron shifts occur radiation is emitted or absorbed according as the new states have less or greater energy than the initial states. With molecules, a greater number of transitions are possible. Molecules, like atoms, have energy because of their revolving electrons; unlike atoms, they have energy, due to their ability to spin about one or more axes of configuration, and also to the fact that their atoms may vibrate with respect to one another. Just as certain electron orbits only are permitted, so only certain velocities of rotation and certain amplitudes of vibration are permitted in these additional methods of storing energy. By placing such limitations upon spins and vibrations all of the characteristic qualities of molecular spectra are explainable. Molecular spectra are band spectra; a typical illustration of this kind of spectra is shown in Fig. 310. It is a much more complicated form of spectrum than the line spectra of atoms, and it is beyond the purpose of this text to discuss them further than to indicate that the same ideas which have successfully solved line or atomic spectra are also effective in the much more complicated molecular spectra.

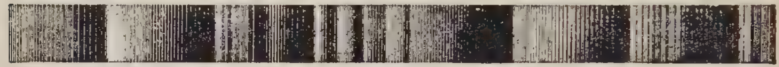


Fig. 310

482. Isotopes.—As was stated in §476, the two materials of which all elements are built are protons and electrons, the positive and the negative elemental electrical charges. As the proton constitutes practically all of the mass of the hydrogen atom, it would appear that the masses of all elements should be integral multiples of the mass of the hydrogen atom. In many instances, however, this is not so, since the atomic weights of the elements lie usually between such integral values. Recently the methods of modern physics have made it possible to deal with individual atoms instead of with the average effect of the enormous number which make up the necessary mass, however small, required by the older atomic weight methods. When measured singly, the atoms of a given element often show a number of atomic masses. These masses are different multiples, always integral, of the mass of the hydrogen atom. This must mean that different numbers of protons are used in forming these atoms. They all have the same nuclear charge and, consequently, the same number of electrons outside the nucleus. Accordingly they have the same physical and chemical properties, since these depend upon the nuclear charge and not upon atomic weight. Atoms which have different numbers of protons and electrons in their nuclei, and consequently different masses, but the same nuclear charge and number of outer electrons, are called *Isotopes.*

The same experimental methods which isolate and weigh the atoms having different masses also determine the relative numbers of each kind found in a given element. From the weights and relative numbers of each kind it is possible to compute the average weight of atoms of the element, and this average value agrees well with the value of the atomic weight obtained by the usual atomic weight determinations. A few of the many elements possessing isotopes are given in the following table.

Element	*Atomic number*	*Atomic weight*	*Number of isotopes*	*Mass numbers of isotopes*
Magnesium	12	24.32	3	24, 25, 26
Sulphur	16	32.06	3	32, 34, 33
Zinc	30	65.38	4	64, 66, 68, 70
Mercury	80	200.6	6	202, 200, 199, 198, 201, 204

Nature builds sulphur in three ways. The first type has 32 protons and 16 electrons in its nucleus, giving a resulting nuclear charge of 16. The second type contains 34 protons and 18 electrons, and the third 33 protons and 17 electrons in their respective nuclei, but in each case the resulting nuclear charge is 16. By keeping this charge constant, Nature has concealed very successfully her lack of regularity in building atoms. Only the methods of modern physics have made it possible to detect these variations.

483. Recent Developments.—The study of atomic and molecular spectra has been one of the main factors leading in the past few years to a tremendous revolution in our basic ideas in physical science, a revolution whose significance is only today beginning to be revealed.

Planck and Bohr led the way in finding experimental evidence showing that the "classical" laws of mechanics and electromagnetism, which we have studied in earlier chapters, are not completely applicable to atomic phenomena. For example, classical theory leads us to expect the electron of the hydrogen atom to fall into the nucleus, which it certainly does not ordinarily do in nature.

The accumulation of such contradictions made atomic physics a more and more embarrassing accumulation of special cases, until in 1925 the time was ripe for a completely new formulation of the laws of mechanics and electromagnetism. Such a new formulation appeared in a rapid succession of contributions by a number of theoretical physicists, notably Schroedinger, Heisenberg, and Dirac, between 1925 and 1927; so that it may now safely be said that the advent of the new theoretical physics, or "quantum mechanics," is generally accepted as a milestone in the advance of our knowledge of nature. It is hardly necessary to state that this acceptance of the quantum mechanics in no way invalidates the well-verified classical laws in their applications to ordinary large-scale phenomena.

An extended treatment of the quantum mechanics would be out of place here. It requires a considerable amount of mathematical groundwork on the part of the student, but a brief sketch of the main contributions of its great pioneers may be profitable.

Schroedinger, following de Broglie, showed (the "wave mechanics") that matter itself can be treated alternatively as an *ensemble* of discrete particles or as a form of continuous wave motion. This is a striking complement to the earlier contribution of Einstein (not his great contribution) on the nature of light, by which it appeared that light might at times be regarded more satisfactorily as a corpuscular than as a wave phenomenon.

Most of the other contributions have been more purely mathematical. Heisenberg laid special stress in the "matrix mechanics" on the fact that physics is a science of *observation*. His equations for the state of an atom contain such quantities as the frequency and amplitude of the light it emits, rather than the position and velocity of an electron in its orbit. Dirac gave a formulation of quantum mechanics, made simpler and more elegant than Heisenberg's by the introduction of a sort of number hitherto familiar in mathematics but not in physics, the most striking property of which is that it does not obey the ordinary commutative law of multiplication, although it obeys all the other ordinary laws of algebra; that is xy is not necessarily equal to yx.

Though the special viewpoints of these and other investigators differ widely, their mathematical results are in complete harmony and lead without any special assumptions to a complete agreement—for instance, with the experimental results for the hydrogen spectrum we have just studied. Moreover, they have been confirmed in such widely differing fields as radioactivity, specific heats, and electrical conductivity, as well as in atomic and molecular spectroscopy. In all these fields the classical theory was either inadequate or led to positively wrong results.

Fields of research ranging from pure experiment to pure mathematics, and even metaphysics, have been opened by the new physics. The frontiers of the science have been so far extended that not for many years will the impetus lent to physical research by the quantum mechanics be lost.

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